

Stabilization of constrained uncertain systems by an off-line approach using zonotopes

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ABSTRACT

In this paper, stabilization of uncertain systems was established using zonotopic sets. The obtained state feedback control laws are computed by an off-line approach reducing computational burdens. The resolution of a robust model predictive control (MPC) allows computing a sequence of state feedback control laws corresponding to a sequence of zonotopic invariant sets. The implemented control laws are then calculated by linear interpolation between the state feedback gains corresponding to the nested pre-computed zonotopic sets. The proposed interpolation with the use of zonotopic sets achieves better control performances.

1 Introduction

Model predictive control (MPC) is one of the most successful techniques of advanced control in the process industry. Thanks to the recent developments of the underlying theoretical framework, MPC has become a mature control technique able to provide controllers ensuring stability, robustness, constraint satisfaction and tractable computation for linear and nonlinear systems [1]. The MPC is can be made in the context of representation in state variables [2]. This not only make use of existing theorems and results in the state space theory, but also facilitates the extension of the theory of model predictive control to more complex cases such as systems with stochastic disturbances, noise on measured variables or multivariable control. For nonlinear uncertain systems, explicitly modeling of the uncertainty is essential [3].

For modeling uncertain systems, it is very important for MPC to be more robust [2]. Important areas in MPC that have recently seen significant theoretical and implementational progress include robust and stochastic MPC as well as efficient computations for MPC via convex and reliable real-time optimization [4].

Although these MPC schemes have remarkable performance and good theoretical properties, there is a hard computational burden due to the minimization of the optimization problem, especially in the presence of the system nonlinearity. The other

is to derive robust stability of MPC by minimization of linear quadratic optimization problems subject to polytopic uncertainty models and linear matrix inequality (LMI) constraints, which was firstly proposed in [5]. From this formulation, a broad class of model uncertainty descriptions can be addressed with guaranteed closed-loop robust stability of MPC.

Since the Lyapunov theory was introduced as an efficient stability analysis tool of systems governed by ordinary differential equations, the notion of set invariant was used in many problems concerning the analysis and control of dynamic systems. An important motivation, leading to introduce invariant sets, was the need to analyze the effect of uncertain systems. An invariant set is a region of the state space such the trajectory generated by the dynamical system remains confined in the set if the initial condition lies within it [6]. Robust controlled invariant set is particularly relevant since it can be used in the context of constrained uncertain systems stability [7].

In recent years, in the theory of control, regardless of a particular area, there have been numerical solutions are extensive. That is, a problem is usually considered as solved whenever it can be written as a (constrained) optimization problem. The difficulty in solving such a problem is greatly influenced by the way the constraint set is defined. In this context, several families of sets vie for influence [8].

Historically, ellipsoidal sets [9] were a useful choice of invariant sets due to their simple definition.

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Then, the problem becomes to design the invariant ellipsoids off-line [10]. Recently, polyhedral sets [11], became widespread due to their representation flexibility and reliable numerical algorithms. Angeli [12] proposed an ellipsoidal off-line MPC scheme for uncertain polytopic systems. In [13] the authors proposed an off-line robust constrained MPC algorithm by choosing a sequence of states.

However, polyhedral sets become numerically unstable for higher dimensions and certain operations scale badly with respect to the complexity of the set in question. Zonotopic sets, a subclass of polyhedral sets [11], have started to gain attention. Their symmetric shape, coupled with the flexibility inherited from the polyhedral class makes them an appealing choice for higher dimensions. Also, for dynamical systems, zonotopes provide an excellent compromise between accuracy and efficiency as first [14]. As a direct consequence, researchers from disparate fields started to employ them in various applications [15,16]. The greater part of this application exploits the zonotope facility in defining robust approximations.

Zonotopes are also used to rigorously estimate the states of dynamical systems as an alternative to observers that optimize with respect to the best estimate, such as Kalman filters. One of the first works that use zonotopes for state-bounding observers is [17] and bounded disturbance in [18]. Similarly to reachability analysis, this work has been extended to nonlinear systems in [19,20] and systems with uncertain parameters [21].

This paper is organised as follows, a description of the considered problem is first presented. Then, the optimal control problem for constrained uncertain systems is formulated. Its resolution procedure using zonotopic invariant sets with an interpolation step, is proposed. The efficiency of the used zonotopic invariant sets is then illustrated by two examples. Finally, the paper is concluded.

2 Problem description

The considered system is the following linear time-varying (LTV) system with polytopic uncertainty:

$$\begin{aligned} x(k+1) &= A(k)x(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \quad (1)$$

where $x(k)$ is the state of the plant, $u(k)$ is the control input and $y(k)$ is the plant output. We assume that:

$$\begin{aligned} [A(k), B(k)] &\in \Omega, \\ \Omega &= \text{conv} \{ [A_1, B_1], [A_2, B_2], \dots, [A_L, B_L] \} \end{aligned} \quad (2)$$

where *conv* is the convex hull and *Omega* is a polytope, $[A_j, B_j]$ are vertices of the polytope such that:

$$[A_j, B_j] = \sum_{j=1}^L \lambda_j [A_j, B_j], \quad \sum_{j=1}^L \lambda_j = 1, \quad 0 \leq \lambda_j \leq 1, \quad (3)$$

The aim of this research is to find a state-feedback control law:

$$u(k+i/k) = Kx(k+i) \quad (4)$$

that stabilizes (1) with the following performance cost:

$$J_\infty(k) = \min_{u(k+i/k)} \max_{[A(k+i), B(k+i)] \in \Omega, i \geq 0} J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix} \quad (5)$$

subject to :

$$|u_h(k+1/k)| \leq u_{h,\max}, \quad h = 1, 2, \dots, n_u \quad (6)$$

$$|y_r(k+1/k)| \leq y_{r,\max}, \quad r = 1, 2, \dots, n_y \quad (7)$$

where $\Theta > 0$ and $R > 0$ are symmetric weighting matrices.

In [13] the authors describe the concept of an asymptotically stable invariant ellipsoid to develop a robust constrained MPC algorithm. This algorithm gives a sequence of explicit control laws corresponding to a sequence of asymptotically stable invariant ellipsoids constructed off-line one within another in state space. They solved, at each time step, the robust constrained MPC problem using Linear Matrix Inequalities (LMI). The obtained result is considered conservative due to invariant ellipsoids which are an approximation of the real invariant sets.

In [5] the authors describe polyhedral invariant sets with an off-line robust algorithm to stabilize uncertain systems. They are calculated off-line a sequence of state feedback control laws corresponding to a sequence of polyhedral invariant sets. At each sampling time, the smallest polyhedral invariant set that the currently measured state can be embedded is determined. The corresponding state feedback control law is then implemented to the process.

We intend to use this algorithms with zonotopic representation of the invariant sets followed by an interpolation step to get less conservative results.

3 Robust MPC Algorithm

In this section, an interpolation-based robust MPC algorithm for uncertain polytopic discrete-time systems using zonotopic invariant sets is presented. The nested zonotopic invariant sets and feedback gains are pre-computed off-line in first step, in order to reduce the on-line computational burdens. In second step, the real-time control law is calculated by linear interpolation between the feedback gains corresponding to the zonotopic invariant sets previously generated. The optimization problem solved at each time step is based on optimization of linear performance index and only a computationally low-demanding optimization problem is required to be solved on-line.

Definition 1: (Invariant sets)

An invariant set is a subset of the state space $\Omega \in R^n$,

such that for all $x_0 \in \Omega$, and all admissible input function $u : R \rightarrow U$, the solution to system (1) with $x(0) = x_0$ satisfies $x(t) \in \Omega$ for all $t \geq 0$.

Intuitively, the system remains trapped in the invariant for all future times [22].

One of the advantages of invariant sets, compared with iterative methods, is that they cover unbounded time horizon, without any extra cost. A second one is that they have in general a compact representation. For example, an invariant ellipsoid is represented by a single nn matrix. Whereas, iterative methods produce a large number of sets, often with growing complexity. Each of these sets has to be taken into account in order to enclose all reachable states.

3.1 Off-line Steps

Step 1: Choose a state sequence $x_i, i \in \{1, 2, \dots, N\}$ and solve the following problem to obtain corresponding state feedback gains:

$$K_i = Y_i Q_i^{-1} \tag{8}$$

The states x_i must be chosen such that the distance between x_{i+1} and the origin is less than the distance between x_i and the origin. Matrices Y_i and Q_i^{-1} , for all $i = 1, 2, \dots, N$ are solutions of the following problem:

$$\min_{\gamma_i, Q_i, Y_i} \gamma_i \tag{9}$$

subject to:

$$\begin{bmatrix} 1 & x_i^T \\ x_i & Q_i \end{bmatrix} \geq 0, \tag{10}$$

$$\begin{bmatrix} Q_i & Q_i A_j^T + Y^T B_j^T & Q_i \Theta^{1/2} Y_i^T R^{1/2} \\ A_j Q_i + B_j Y_i & Q_i & 0 & 0 \\ \Theta^{1/2} Q_i & 0 & \gamma_i I & 0 \\ R^{1/2} Y_i & 0 & 0 & \gamma_i I \end{bmatrix} \geq 0 \tag{11}$$

$\forall j = 1, 2, \dots, L$

$$\begin{bmatrix} X & Y_i \\ Y_i^T & Q_i \end{bmatrix} \geq 0, X_{hh} \leq u_{h, \max}^2, h = 1, 2, \dots, n_u \tag{12}$$

$$\begin{bmatrix} S & C(A_j Q_i + B_j Y_i) \\ (A_j Q_i + B_j Y_i)^T C^T & Q_i \end{bmatrix} \geq 0, S_{rr} \leq y_{r, \max}^2, \tag{13}$$

$r = 1, 2, \dots, n_y, \forall j = 1, 2, \dots, L,$

Step 2: Given the state feedback gains:

$$K_i = Y_i Q_i^{-1}, i \in \{1, 2, \dots, N\} \tag{14}$$

from step 1. For each K_i , the corresponding polyhedral invariant sets defined by:

$$S_i = \{x_i / M_i x_i \leq d_i\} \tag{15}$$

are constructed by the following :

Step 2.1: Set $M_i = [C^T, -C^T, K_i^T, -K_i^T]^T, d_i = [y_{\max}^T, y_{\min}^T, u_{\max}^T, u_{\min}^T]^T$ and $m = 1$.

Step 2.2: Select row m from (M_i, d_i) and check

whether $M_{i,m}(A_j + B_j K_i)x \leq d_{i,m}$ is redundant with respect to the constraints defined by (M_i, d_i) by solving the problem:

$$\max_x W_{i,m,j} \tag{16}$$

subject to

$$W_{i,m,j} = M_{i,m}(A_j + B_j K_i)x - d_{i,m}, \quad M_i x \leq d_i \tag{17}$$

Step 2.3: Let $m = m + 1$ and return to Step 2.2. If m is strictly larger than the number of rows in (M_i, d_i) then terminate.

3.2 On-line Step using polyhedral sets

3.2.1 Without interpolation

At each sampling time, determine the smallest polyhedral invariant set $S_i = \{x_i / M_i x_i \leq d_i\}$ where $i = 1, 2, \dots, N - 1$.

containing the measured states and implement the corresponding state feedback control law $u(k/k) = K_i x(k/k)$ to the process.

3.2.2 With 3-points interpolation

At each sampling time, if the measured state lies between S_i, S_{i+1} and $S_{i+2}, i = 1, 2, \dots, N - 1$ implement the interpolated gain obtained by :

$$K = \alpha_1 K_{i-2} + \alpha_2 K_{i-1} + \alpha_3 K_i \tag{18}$$

where $0 < \alpha_i < 1$, for all $i = 1, 2, 3$ and $\sum_{i=1}^3 \alpha_i = 1$.

3.3 On-line Step using zonotopic sets

Zonotopes are convex polytopes that are centrally symmetric. Equivalently, a zonotope is a Minkowski sum of a finite set of line segments. A polytope is a zonotope if it can also be represented by so-called generators (G-representation).

Definition 2: (G-representation of a zonotope)

Given a vector $c \in R^n$ and a set of vectors of $R^n, G = \{g_1, \dots, g_m\}, m \geq n$, a zonotope Z of order m is defined as following:

$$Z = \left\{ x \in R^n, x = c + \sum_{i=1}^p \gamma_i \cdot g_i; -1 \leq \gamma_i \leq 1 \right\} \tag{19}$$

The vector c is called the center of the zonotope Z . The vectors g_1, \dots, g_m are called generators of Z .

The order of zonotope is defined by the number of its generators (m in this case). In the case of $m < n$, its called degenerated zonotope.

This definition is equivalent with the definition of zonotopes by the Minskowski sum of a finite number of line segments defined by $g_i B^1$. $Z = (c; g_1, g_2, \dots, g_m) = c \oplus g_1 B^1 \oplus \dots \oplus g_m B^1$ Where B^1 is a unitary box in R^n , is a box composed by n unitary intervals. And \oplus is the Minkowski sum.

Definition 3: (Minkowski sum)

The Minkowski sum of two sets X and Y is defined by $X \oplus Y = \{x + y : x \in X, y \in Y\}$.

Definition 4: (Unitary interval)

The unitary interval is defined by $B^n = [-1, 1]$.

Definition 5: (Box)

A box is an interval vector. An interval hull of a set $Z \subseteq R^n$, denoted by $\diamond Z$ is a box that satisfies $Z \subseteq \diamond Z$

Given a box $\diamond Z = ([a_1, b_1], \dots, [a_n, b_n])^T$. $mid(\diamond Z)$, denotes its center and $diam(\diamond Z) = (b_1 - a_1, \dots, b_n - a_n)^T$

Definition 6: (Unitary box)

A unitary box, denoted by B^n is a box compound by n unitary intervals.

Definition 7: (V-representation of a polytope)

Given r vertices $v_i \in R^n$, $P = conv\{v_1, \dots, v_r\}$ is a convex polytope, where $conv$ is the convex hull operator.

To obtain zonotopic sets from polyhedral ones, we have to perform the following three steps:

Step 1: Compute the vertices $v_i \in R^n$ (V-representation) of all N polytopes $S_i, i = 1, \dots, N$.

Step 2: Obtain the minimum and maximum values of each polytope i :

$$\begin{aligned} m_{\min} &= \min(V_i^1, \dots, V_i^r), \\ m_{\max} &= \max(V_i^1, \dots, V_i^r). \end{aligned} \quad (20)$$

where V_i^j is the i th component of v^j and r is the number of the vertices of each polytope.

Step 3: Compute a G-representation of the n -dimensional interval $[m_{\min}, m_{\max}]$:

$$[m_{\min}, m_{\max}] = \left\{ x = c + \sum_{i=1}^n \gamma_i \cdot g_i, -1 \leq \gamma_i \leq 1 \right\}, \quad (21)$$

where :

$$c = 0.5(m_{\min} + m_{\max}), \quad (22)$$

$$g_i^{(i)} = \begin{cases} 0.5(m_{\max} - m_{\min}), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \quad (23)$$

3.3.1 Without interpolation

At each sampling time, determine the smallest invariant zonotope

$$Z = \left\{ x | x = c + \sum_{i=1}^p \gamma_i \cdot g_i, -1 \leq \gamma_i \leq 1 \right\}, i = 1, 2, \dots, N - 1$$

containing the measured states and implement the corresponding state feedback control law $u(k/k) = K_i x(k/k)$ to the process.

3.3.2 With 2-points interpolation

At each sampling time, if the measured state lies between Z_i and Z_{i-1} , implement the interpolated gain obtained by :

$$K = \alpha K_i + (1 - \alpha) K_{i+1} \quad (24)$$

where $0 < \alpha_i < 1$, for all $i = 1, 2$, and $\sum_{i=1}^2 \alpha_i = 1$.

3.3.3 with 3-points interpolation

At each sampling time, if the measured state lies between Z_i, Z_{i-1} and Z_{i-2} , implement the interpolated gain obtained by:

$$K = \alpha_1 K_{i-2} + \alpha_2 K_{i-1} + \alpha_3 K_i \quad (25)$$

where $0 < \alpha_i < 1$, for all $i = 1, 2, 3$, and $\sum_{i=1}^3 \alpha_i = 1$.

4 Application

In this section, we are going to present two examples allowing to implement the proposed approach. For both examples, the software Yalmip toolbox [23] in the MATLAB environment was used to compute the solution of the LMI minimization problem.

4.1 Example 1

Lets consider an uncertain non-isothermal CSTR [5] where the exothermic reaction AB takes place. The reaction is irreversible and the rate of reaction is first order with respect to component A. A cooling coil is used to remove heat that is released in the exothermic reaction. The uncertain parameters are: the reaction rate constant k_0 and the heat of reaction ΔH_{rxn} . The linearized model based on the component balance and the energy balance is given by the following state equations:

$$\begin{cases} x(t+1) = A(t)x(t) + B(t)u(t) \\ y(t) = Cx(t) \end{cases} \quad (26)$$

where $x = \begin{bmatrix} C_A \\ T \end{bmatrix}$ is the state vector $x(t)$ and $u =$

$\begin{bmatrix} C_{A,F} \\ F_C \end{bmatrix}$ is the control input vector $u(t)$. Matrices are defined by:

$$A = \begin{pmatrix} -\frac{F}{V} - k_0 e^{-E/RT_s} & -\frac{E}{RT_s^2} k_0 e^{-E/RT_s} C_{A_s} \\ \frac{-\Delta H_{rxn} k_0 e^{-E/RT_s}}{\rho C_p} & -\frac{F}{V} - \frac{UA}{V \rho C_p} - \Delta H_{rxn} \frac{E}{\rho C_p RT_s^2} k_0 e^{-E/RT_s} C_{A_s} \end{pmatrix} \quad (27)$$

$$B = \begin{bmatrix} \frac{F}{V} & 0 \\ 0 & -2.098 \times 10^5 \frac{T_s - 365}{V \rho C_p} \end{bmatrix}, \quad (28)$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (29)$$

Where C_A is the concentration of A in the reactor, $C_{(A,F)}$ is the feed concentration of A, T is the reactor temperature, and F_C is the coolant flow. The operating parameters are: $F = 1 \text{ m}^3/\text{min}$, $V = 1 \text{ m}^3$, $k_0 = 10^9 - 10^{10} \text{ min}^{-1}$, $\frac{E}{R} = 8330.1 \text{ K}$, $-\Delta H_{rxn} = 10^7 - 10^8 \text{ cal/kmol}$, $\rho = 10^6 \text{ g/m}^3$, $UA = 5.34 \times 10^6 \text{ cal/(k min)}$ and $C_p = 1 \text{ cal/(gk)}$. Let $\bar{C}_A = C_A - C_{A,eq}$, $\bar{T}_A = T - T_{eq}$, $\bar{C}_{A,F} = C_{A,F} - C_{A,F,eq}$ and $\bar{F}_C = F_C - F_{C,eq}$ where the subscript eq is used to denote the corresponding variable at equilibrium condition. By discretization, using a sampling time of 0.15 min, the

discrete-time model with $\begin{bmatrix} \bar{C}_A(k) \\ \bar{T}(k) \end{bmatrix}$ as a state vector and $\begin{bmatrix} \bar{C}_{A,F}(k) \\ \bar{F}_C(k) \end{bmatrix}$ as a control vector, is given as follows:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = Cx(k) \end{cases} \quad (30)$$

where:

$$A = \begin{bmatrix} 0.85 - 0.0986\alpha(k) & -0.0014\alpha(k) \\ 0.9864\alpha(k)\beta(k) & 0.0487 + 0.01403\alpha(k)\beta(k) \end{bmatrix}$$

$$B = \begin{bmatrix} 0.15 & 0 \\ 0 & -0.912 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (31)$$

where: $1 \leq \alpha(k) = k_0/10^9 \leq 10$ and $1 \leq \beta(k) = -\Delta H_{rxn}/10^7 \leq 10$.

The two parameters (k) and (k) are independent of each other. Then, we consider the following polytopic uncertain model with four vertices:

$$\Omega = Co \left\{ \begin{bmatrix} 0.751 & -0.0014 \\ 0.986 & 0.063 \end{bmatrix}, \begin{bmatrix} 0.751 & -0.0014 \\ 9.864 & 0.189 \end{bmatrix}, \begin{bmatrix} 0.751 & -0.0014 \\ 0.986 & 0.063 \end{bmatrix}, \begin{bmatrix} 0.751 & -0.0014 \\ 9.864 & 0.189 \end{bmatrix} \right\} \quad (32)$$

The objective is to regulate the concentration \bar{C}_A and the reactor temperature \bar{T} to the origin by manipulating $\bar{C}_{A,F}$ and \bar{F}_C , respectively. These variables are constrained by: $|\bar{C}_{A,F}| \leq 0.5 \text{ kmol/m}^3$, and $|\bar{F}_C| \leq 1.5 \text{ m}^3/\text{min}$.

The cost function is given by (5) with $\Theta = I$ and $R = 0.1I$.

Lets choose a sequence of states:

$$x_i = \left\{ \begin{matrix} (0.0525, 0.0525), (0.0475, 0.0475), \\ (0.0425, 0.0425), (0.0375, 0.0375), \\ (0.0325, 0.0325), (0.0275, 0.0275) \end{matrix} \right\} \quad (33)$$

This sequence is used to calculate six off-line feedback gains $K_i, i = 1, 2, \dots, 6$. The regulated output (the concentration of A and the reactor temperature), when $\alpha(k)$ and $\beta(k)$ are randomly time-varying between $10^9 \leq \alpha(k) = 10^{10}$ and $10^7 \leq \beta(k) = \Delta H_{rxn} \leq 10^8$.

The obtained zonotopes are defined by:

$$c_i = \{ 2.98, 3.17, -1.31, 1.31, -3.17, -2.98 \}, \quad (34)$$

Where c_i is the center of the zonotope $Z_i, i = 1, 2, \dots, 6$.

The generators matrices are defined by:

$$g_i = \begin{pmatrix} 3.07 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.26 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.29 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.26 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.07 \end{pmatrix} \quad (35)$$

for all $i = 1, 2, \dots, 6$.

The regulated outputs are shown respectively in Figure 1 and Figure 2. It is seen that the considered zonotopic sets give less conservative results and better system performance as compared to the approach using polyhedral ones.

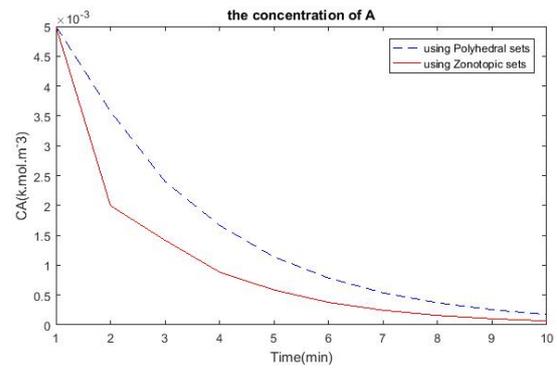


Figure 1: The concentration of A in the reactor of the

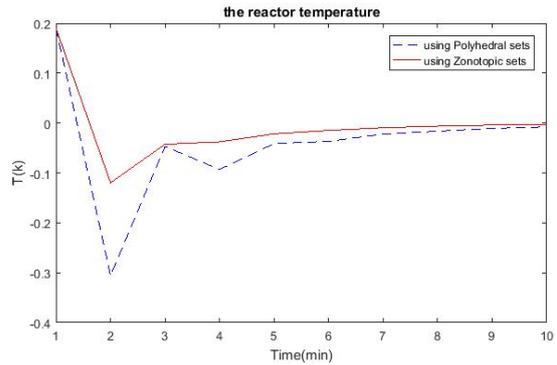


Figure 2: The reactor temperature of the regulated output.

In Figure 3 and Figure 4 respectively, it is seen that the considered interpolation using three zonotopic sets, give less conservative results as compared to the approach with interpolation of polyhedral sets.

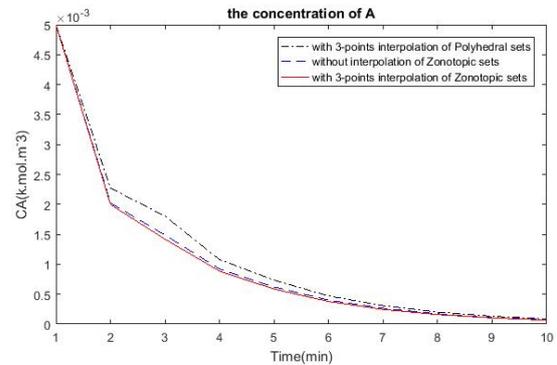


Figure 3: The concentration of A in the reactor of the regulated output

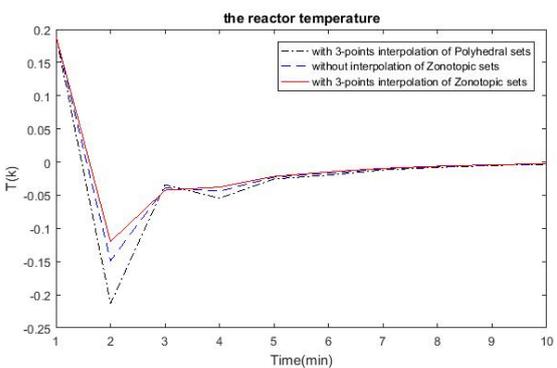


Figure 4: The reactor temperature of the regulated output

4.2 Example 2

We consider the angular positioning system [4]. It consists of an electric motor driving a rotating antenna so that it always points in the direction of a moving object. The motion of the antenna can be described by the following discrete time-equation:

$$\begin{cases} \begin{bmatrix} \theta(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\alpha(k) \end{bmatrix} \begin{bmatrix} \theta(k) \\ \dot{\theta}(k) \end{bmatrix} \\ \quad + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k) \\ y(k) = [1 \ 0] \begin{bmatrix} \theta(k) \\ \dot{\theta}(k) \end{bmatrix} \end{cases} \quad (36)$$

where $\theta(k)$ is the angular position of the antenna, $\dot{\theta}(k)$ is the angular velocity and $u(k)$ is the input voltage of the motor. It is assumed that the uncertain parameter is arbitrarily time-varying : $0.1\alpha(k)10$.

Let $\bar{\theta} = \theta - \theta_{eq}$, $\bar{\dot{\theta}} = \dot{\theta} - \dot{\theta}_{eq}$ and $\bar{u} = u - u_{eq}$ where the subscript eq denotes the corresponding variable at equilibrium condition. The obtained system can be written as follows:

$$\begin{cases} \begin{bmatrix} \bar{\theta}(k+1) \\ \bar{\dot{\theta}}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\alpha(k) \end{bmatrix} \begin{bmatrix} \bar{\theta}(k) \\ \bar{\dot{\theta}}(k) \end{bmatrix} \\ \quad + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k) \\ y(k) = [1 \ 0] \begin{bmatrix} \bar{\theta}(k) \\ \bar{\dot{\theta}}(k) \end{bmatrix} \end{cases} \quad (37)$$

The system (36) has the following polytopic structure:

$$A(k) \in conv \left\{ \begin{bmatrix} 1 & 0.1 \\ 0 & 0.9 \end{bmatrix}, \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix} \right\} \quad (38)$$

The input constraint is:

$$|\bar{u}(k)| \leq 2 \text{ volts} \quad (39)$$

The weighting matrices Θ and R are given by:

$$\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } R = 0.00002I \quad (40)$$

Lets choose the following sequence of seven states:

$$x_i = \left\{ \begin{matrix} (0.35, 0.35), (0.3, 0.3), \\ (0.25, 0.25), (0.02, 0.02), \\ (0.15, 0.15), (0.1, 0.1), (0.05, 0.05) \end{matrix} \right\} \quad (41)$$

This sequence is used to calculate seven state feedback gains K_i corresponding to seven polyhedral invariant sets. The obtained zonotopes are defined by their centers:

$$c_i = \left\{ \begin{matrix} 1.52, -0.08, -0.21, 0.21, \\ 0.08, -1.52, 0.21 \end{matrix} \right\} i = 1, 2, \dots, 7. \quad (42)$$

The zonotope generators are given by:

$$g_i = \begin{pmatrix} 0.82 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.87 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.70 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.38 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.59 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.87 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.47 \end{pmatrix} \quad (43)$$

for all $i = 1, 2, \dots, 7$.

Figure 5 and Figure 6 represent closed-loop responses of the system when $\alpha(k)$ is randomly time-varying between $0.1 \leq \alpha(k) \leq 10$.

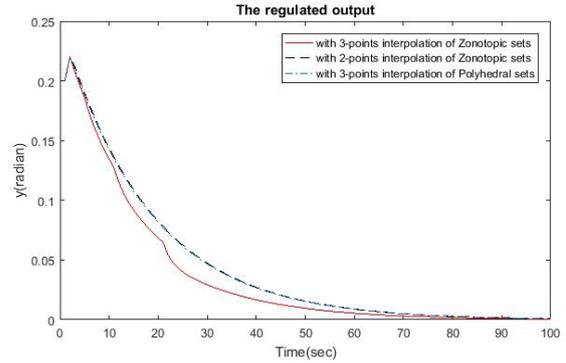


Figure 5: The regulated output

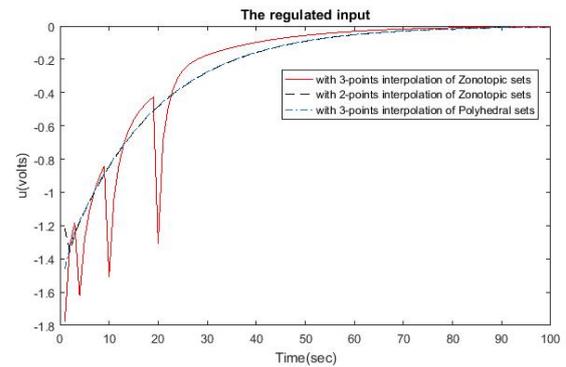


Figure 6: The control input

We can observe that by the considered approach with zonotopic sets using three points interpolation especially the one with three control gains, we obtain better control performances as compared to the approach with interpolation of polyhedral sets.

5 Conclusion

In this paper, we have presented an input feedback robust model predictive control of polytopic uncertain discrete-time systems. The proposed algorithm used an off-line optimal control optimization problems solution to determine a sequence of feedback gains. A sequence of nested zonotopic invariant sets associated with pre-computed feedback gains are constructed. At each control iteration, the smallest invariant containing the measured states is identified, and the corresponding feedback gain is implemented. In

addition, an interpolation step to the obtained control laws based on polyhedral and zonotopic invariant sets respectively was employed. The proposed approach applied on examples showed that the control performance using zonotopic invariant sets followed by an interpolation of the nested zonotopes is better than the one using polyhedral invariant sets.

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