Degenerate $p(x)$-elliptic equation with second membre in $L^1$

Adil Abbassi1, Elhoussine Azroul2, Abdelkrim Barbara2

1 Sultan Moulay Slimane University, Mathematics, LMACS Laboratory, FST, Beni-Mellal, Morocco
2 Sidi Mohamed Ben Abdellah University, Mathematics, LAMA Laboratory, FSDM, Fez, Morocco

ARTICLE INFO

Article history:
Received: 12 April, 2017
Accepted: 04 May, 2017
Online: 10 December, 2017

Keywords:
Sobolev spaces with weight and to variable exponents
Truncations

ABSTRACT

In this paper, we prove the existence of a solution of the strongly nonlinear degenerate $p(x)$-elliptic equation of type:

$$\begin{cases}
-\text{div} a(x,u,\nabla u) + g(x,u,\nabla u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $a$ is a Carathéodory function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}^N$, which satisfies assumptions of growth, ellipticity and strict monotonicity. The nonlinear term $g: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ checks assumptions of growth, sign condition and coercivity condition, while the right hand side $f$ belongs to $L^1(\Omega)$.

1 Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, let $\partial \Omega$ its boundary and $p(x) \in C(\overline{\Omega})$ with $p(x) > 1$.

Let $\nu$ be a weight function in $\Omega$, i.e. $\nu$ measurable and strictly positive a.e. in $\Omega$. We suppose furthermore, that the weight function satisfies also the integrability conditions defined in section 2.

Let us consider the following degenerate $p(x)$-elliptic problem with boundary condition

$$\begin{cases}
Au + g(x,u,\nabla u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $A$ is a Carathéodory function from $W^{1,p(x)}(\Omega,\nu)$ to its dual $W^{-1,p'(x)}(\Omega,\nu^*)$, with $\nu^* = \nu^{1-p'(x)}$.

Let consider the following degenerate $p(x)$-elliptic problem of Dirichlet:

$$\begin{cases}
Au + g(x,u,\nabla u) = f & \text{in } D'(\Omega), \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

In the case where $p$ is constant and without weight, there is a wide literature in which one can find existence results for problem [1]. When the second member $f$ belongs to $W^{-1,p'}(\Omega)$, A. Bensoussan, L. Boccardo and F. Murat [5] studied the problem and gave an existence result. While if $f \in L^1(\Omega)$ the initiated basic works were given by H. Brezis and Strauss [3], L. Boccardo and T. Gallouët [7] also proved an existence result for [1], which was extended to the a unilateral case studied by A. Benkirane and A. Elmahi [5].

When $g$ is not necessarily the null function, T. Del Vecchio [10] proved first existence result for problem [1] in the case where $g$ does not depend on the gradient and then in V. M. Monetti and L. Randazzo [16] using, in both works, the rearrangement techniques.

Whoever in [1], Y. Akdim, E. Azroul and A. Benkirane treated the problem [1] within the framework of Sobolev spaces with weight $W^{1,p}(\Omega,\omega)$, but while keeping $p$ constant.

E. Azroul, A. Barbara and H. Hjiej [2] studied [1], in the nonclassical case by considering nonstandard Sobolev spaces without weight $W^{1,p(x)}(\Omega)$. See as well [3] where existence and regularity of entropy solutions was obtained for equation [1] with degenerated...
second member. Our objective, in this paper, is to study equation \(1\) by adopting Sobolev spaces with weight \(v(x)\), and to variable exponents \(p(x)\), \(W_{0}^{1,p(x)}(\Omega, \nu)\). We prove that the problem \(1\) admits at least a solution \(u \in W_{0}^{1,p(x)}(\Omega, \nu)\).

2 Functional frame

Throughout this section, we suppose that the variable exponent \(p(\cdot) : \Omega \to [1, +\infty]^{\max}\) is log-Hölder continuous on \(\Omega\), that is there is a real constant \(c > 0\) such that \(\forall x, y \in \Omega, x \neq y\) with \(|x - y| < 1/2\) one has:

\[
|p(x) - p(y)| \leq \frac{c}{-\log|x - y|}
\]

and satisfying

\[
p^{-} \leq p(x) \leq p^{+} < +\infty
\]

where

\[
p^{-} := \text{ess inf}_{x \in \Omega} p(x); \quad p^{+} := \text{ess sup}_{x \in \Omega} p(x).
\]

We define

\[
C_{+}(\Omega) = \{h \text{ log-Hölder continuous on } \overline{\Omega}, h(x) > 1\}.
\]

Definition 1 Let \(v\) be a function defined in \(\Omega\); we call \(v\) a weight function in \(\Omega\) if it is measurable and strictly positive a.e. in \(\Omega\).

2.1 Lebesgue spaces with weight and to variable exponents

Let \(p \in C_{+}(\overline{\Omega})\) and \(v\) be a weighted function in \(\Omega\). We define the Lebesgue space with weight and to variable exponents \(L^{p(x)}(\Omega, \nu)\), by

\[
L^{p(x)}(\Omega, \nu) = \{u : \Omega \to \IR, \text{ measurable} : \int_{\Omega} v(x)|u|^{p(x)}dx < \infty\},
\]

equipped with the Luxemburg norm:

\[
\|u\|_{p(x), \nu} = \inf\{\mu > 0 : \int_{\Omega} v(x)\frac{|u|^{p(x)}}{\mu}dx \leq 1\}
\]

Proposition 1 The space \(\left(L^{p(x)}(\Omega, \nu), \|\cdot\|_{p(x), \nu}\right)\) is of Banach.

Proof: Considering the operator

\[
M_{\nu}^{1/p(x)} : L^{p(x)}(\Omega, \nu) \to L^{p(x)}(\Omega), f \mapsto \nu^{1/p(x)} f = f_{\nu}^{1/p(x)}
\]

It’s clear that \(M_{\nu}^{1/p(x)}\) is isomorphism from \(L^{p(x)}(\Omega, \nu)\) into \(L^{p(x)}(\Omega)\), then \(M_{\nu}^{1/p(x)}\) is a continuous biuniformly application. Seeing that \(L^{p(x)}(\Omega)\) is a Banach space, then \(\left(L^{p(x)}(\Omega, \nu), \|\cdot\|_{p(x), \nu}\right)\) is of Banach.

Let's note \(\rho_{\nu}(u) = \int_{\Omega} v(x)|u|^{p(x)}dx\).

Remark 1 In simple case \(v(x) = 1\), we find again the Lebesgue space with variable exponents \(L^{p(x)}(\Omega)\); and \(\rho_{\nu}(u) = \rho(\nu) := \int_{\Omega} |u|^{p(x)}dx\), (see \([12],[13]\) and \([17]\)).

Lemma 1 For all function \(u \in L^{p(x)}(\Omega, \nu)\). There are the following assertions:

(i) \(\rho_{\nu}(u) > 1\) \(\Rightarrow\) \(||u||_{p(x), \nu} > 1\) \(\Rightarrow\) \(||u||_{p(x), \nu} \geq 1\) \(\Rightarrow\).

(ii) If \(||u||_{p(x), \nu} > 1\) then \(\|u\|^{p(x)}_{p(x), \nu} \leq \rho_{\nu}(u) \leq \|u\|^{p(x)}_{p(x), \nu} - 1\).

(iii) If \(||u||_{p(x), \nu} < 1\) then \(\|u\|^{p(x)}_{p(x), \nu} \leq \rho_{\nu}(u) \leq \|u\|^{p(x)}_{p(x), \nu} - 1\).

Proof: Seeing that \(\rho_{\nu}(u) = \rho(u^{p(x)}\nu^{1/p(x)})\) and \(\|v^{1/p(x)}\nu\|_{p(x)} = \|v\|^{1/p(x)}_{p(x)}\), we prove the lemma 2.1 above.

Let \(v\) be a weight function such that the following condition:

\[
(w1) \varepsilon L^{1}_{\text{loc}}(\Omega); \quad v^{1/p(x)} \in L^{1}_{\text{loc}}(\Omega).
\]

Proposition 2 Let \(\Omega\) be a bounded open subset of \(\IR^{N}\), and \(v\) be a weight function on \(\Omega\). If \((w1)\) is verified then \(L^{p(x)}(\Omega, \nu) \hookrightarrow L^{1}_{\text{loc}}(\Omega)\).

Proof: Let \(K\) be a included compact on \(\Omega\). Using Hölder inequality we have

\[
\int_{K} |u|dx = \int_{K} |u|^{\frac{1}{p(x)}} v^{\frac{1}{p(x)}}dx \leq 2 \|u\|^{\frac{1}{p(x)}}_{L^{p(x)}(K)} \|v^{\frac{1}{p(x)}}_{L^{p(x)}(K)}\|_{L^{\frac{1}{p(x)}}(\Omega, \nu)}.
\]

Thanks to the assumption \((w1)\) we deduce that

\[
\int_{K} |u|dx \leq C\|u\|_{p(x), \nu}.
\]

2.2 Spaces of Sobolev with weight and to variable exponents

Let \(v \in C_{+}(\overline{\Omega})\) and \(v\) be a weight function in \(\Omega\). We define the space of Sobolev with weight and to variable exponents denoted \(W^{1,p(x)}(\Omega, \nu)\), by

\[
W^{1,p(x)}(\Omega, \nu) = \left\{u \in L^{p(x)}(\Omega, \nu) : \frac{\partial u}{\partial x_{i}} \in L^{p(x)}(\Omega, \nu), i = 1, ..., N\right\},
\]

equipped with the norm

\[
\|u\|_{1,p(x), \nu} = \|u\|_{p(x), \nu} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_{i}}\|_{p(x), \nu}.
\]
which is equivalent to the Luxemburg norm
\[ \| \mu \| = \inf \left\{ \mu > 0 : \int_{\Omega} \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \left( \frac{\partial u}{\partial x_i} \right) dx \leq 1 \right\} \]

**Proposition 3** Let \( \nu \) be a weight function in \( \Omega \) who checks the condition (w1). Then the space \( \left( W^{1,p(x)}(\Omega, \nu), \| \cdot \|_{1,p(x),\nu} \right) \) is of Banach.

**Proof:**
Let us consider \( (u_n)_n \) a Cauchy sequence of \( \left( W^{1,p(x)}(\Omega, \nu), \| \cdot \|_{1,p(x),\nu} \right) \).
Then \( (u_n)_n \) is a Cauchy sequence of \( L^{p(x)}(\Omega) \) and the sequence \( \left( \frac{\partial u_n}{\partial x_i} \right)_n \) is also a Cauchy sequence \( L^{p(x)}(\Omega, \nu), i = 1,...,N \).
According to the proposition 2.1, there exists \( u \in L^{p(x)}(\Omega) \) such that \( u_n \to u \) in \( L^{p(x)}(\Omega) \), and there exists \( v_i \in L^{p(x)}(\Omega, \nu) \) such that \( \frac{\partial u_n}{\partial x_i} \to v_i \) in \( L^{p(x)}(\Omega, \nu), i = 1,...,N \).

Seeing that proposition 2.2, we have \( L^{p(x)}(\Omega, \nu) \subset L^{1,\infty}(\Omega) \) and \( L^{1,\infty}(\Omega) \subset D'(\Omega) \).
Thus, we obtain \( \forall \varphi \in D(\Omega), \)
\[
\langle T_{v_i}, \varphi \rangle = \lim_{n \to \infty} \langle T_{u_n}, \varphi \rangle,
\]
\[
= -\lim_{n \to \infty} \langle T_{u_n}, \frac{\partial u_n}{\partial x_i} \rangle,
\]
\[
= -(T_{u}, \frac{\partial \varphi}{\partial x_i}),(T_{\frac{\partial u}{\partial x_i}}, \varphi).
\]

Hence \( T_{v_i} = T_{\frac{\partial u}{\partial x_i}} \), i.e. \( v_i = \frac{\partial u}{\partial x_i} \).

Consequently \( u \in W^{1,p(x)}(\Omega, \nu), \)
and \( u_n \to u \) in \( W^{1,p(x)}(\Omega, \nu). \)

We deduce then that \( W^{1,p(x)}(\Omega, \nu) \) is a complete space.

On another side, seeing that \( \nu \) satisfies the condition (w1), we prove that \( C^{0,\infty}(\Omega) \) is included in \( W^{1,p(x)}(\Omega, \nu) \); that enables us to define the following space
\[
W^{1,p(x)}_0(\Omega, \nu) = C^{0,\infty}(\Omega) \\| \cdot \|_{1,p(x),\nu},
\]
who is closed in complete space, then it’s complete.

**Proposition 4** (Characterization of the dual space \( (W^{1,p(x)}_0(\Omega, \nu))^* \))
Let \( p(.) \in C_+(\Omega) \) and \( \nu \) a vector of weight who satisfies the condition (w1). Then for all \( G \in (W^{1,p(x)}_0(\Omega, \nu))^* \), there exist a unique system of functions \( (g_0, g_1, ..., g_N) \in L^{p(x)}(\Omega) \times (L^{p(x)}(\Omega, \nu^{1/p(x)}))^N \) such that \( \forall f \in W^{1,p(x)}_0(\Omega, \nu) \):
\[
G(f) = \int_{\Omega} f(x) g_0(x) dx + \sum_{i=1}^{N} \int_{\Omega} \frac{\partial f}{\partial x_i} g_i(x) dx.
\]

**Proof:**
The proof of this proposition is similar to that of theorem 3.16.

Besides the (w1) assumption, we suppose that the function weight satisfied
\[
(w2) \quad \nu^{-s(x)} \in L^{1,\infty}(\Omega)
\]
where \( s \) is a positive function to specify afterwards.
Let us introduce the function \( p_s \) defined by
\[
p_s(x) = \frac{p(x)s(x)}{s(x) + 1}
\]
we have \( p_s(x) < p(x) \) a.e. in \( \Omega \).

and \[
\begin{cases}
  p_s^+(x) = \frac{Np_s(x)}{Np_s(x) - 1} & \text{if } p(x)s(x) < N(s(x) + 1), \\
  p_s^+(x) \text{ arbitrary} & \text{else if,}
\end{cases}
\]

**Proposition 5** Let \( p,s \in C_+(\Omega) \) and \( \nu \) a function weight which satisfies (w1) and (w2). Then \( W^{1,p(x)}(\Omega, \nu) \hookrightarrow W^{1,p(x)}(\Omega, \nu) \).

**Proof:**
According to the Hölder inequality, we have
\[
\int_{\Omega} |v(x)p_s(x) dx = \int_{\Omega} |v(x)|^{p_s(x)} p_s(x) \frac{p(x)}{p_s(x)} \frac{p_s(x)}{p(x)} dx 
\]
\[
\leq \left( \frac{1}{(\frac{N}{p_s(x)})^\frac{1}{p(x)} + \frac{1}{(\frac{N}{p_s(x)})^\frac{1}{s(x) + 1}}} \right) \int_{\Omega} |v(x)|^{p_s(x)} p_s(x) \frac{p(x)}{p_s(x)} \frac{p_s(x)}{p(x)} dx 
\]
\[
\leq C_0 \left( \int_{\Omega} |v(x)|^{p_s(x)} p_s(x) dx \right)^\frac{1}{p(x)} \left( \int_{\Omega} |v(x)|^{-s(x)} dx \right)^\frac{1}{s(x) + 1} 
\]
\[
\leq C_0 C_1 \left( \int_{\Omega} |v(x)|^{p_s(x)} p_s(x) dx \right)^\frac{1}{p(x)} \left( \int_{\Omega} |v(x)|^{-s(x)} dx \right)^\frac{1}{s(x) + 1}, \text{ according to (w2).}
\]

If we take \( v = \frac{\partial u}{\partial x_i} \), we will obtain then
\[
\int_{\Omega} |\frac{\partial u}{\partial x_i}|^{p(x)} dx \leq C_0 C_1 \left( \int_{\Omega} |\frac{\partial u}{\partial x_i}|^{p(x)} \nu(x) dx \right)^\frac{1}{p(x)}
\]
where
\[
\gamma_1 = \begin{cases}
  \left( \frac{p(x)}{p_s(x)} \right)^- & \text{if } \|v(x)|^{p_s(x)} p_s(x) \frac{p(x)}{p_s(x)} \|_{p(x)} \geq 1, \\
  \left( \frac{p(x)}{p_s(x)} \right)^+ & \text{if } \|v(x)|^{p_s(x)} p_s(x) \frac{p(x)}{p_s(x)} \|_{p(x)} < 1,
\end{cases}
\]

consequently
\[
\|\frac{\partial u}{\partial x_i}(x)\|_{p(x),\nu}^\frac{1}{p(x)} \leq C_0 C_1 \left( \int_{\Omega} |\frac{\partial u}{\partial x_i}|^{p(x)} \nu(x) dx \right)^\frac{1}{p(x)}
\]
\[
\leq C_0 C_1 \left( \frac{1}{\gamma_1(x)_{p(x),\nu}} \right)^\frac{1}{p(x)}
\]
where
\[
\gamma_2 = \begin{cases}
  \left( \frac{p(x)}{p_s(x)} \right)_- & \text{if } \|\frac{\partial u}{\partial x_i}(x)\|_{p(x),\nu} \geq 1, \\
  \left( \frac{p(x)}{p_s(x)} \right)_+ & \text{if } \|\frac{\partial u}{\partial x_i}(x)\|_{p(x),\nu} < 1,
\end{cases}
\]
and
\[
\gamma_3 = \begin{cases} 
p^+ & \text{if } \|\frac{\partial u}{\partial x_i}\|_{p(x),\nu} \geq 1, \\
p^- & \text{if } \|\frac{\partial u}{\partial x_i}\|_{p(x),\nu} < 1.
\end{cases}
\]
thus
\[
\|\frac{\partial u}{\partial x_i}\|_{p(x),\nu} \leq C_0 C_1 \|\frac{\partial u}{\partial x_i}\|^{\frac{1}{p(x)}}_{p(x),\nu}, \quad i = 1, 2, \ldots, N. \tag{2}
\]
Seeing that
\[p_s(x) < p(x) \text{ a.e. in } \Omega.\]
then, there is a constant
\[C > 0 \text{ such that } \|u\|_{L^{p(x)}(\Omega)} \leq C\|u\|_{L^{p(x)}(\Omega)},\]
we conclude then that
\[W^{1,p(x)}(\Omega, \nu) \hookrightarrow W^{1,p_s(x)}(\Omega).\]

3 Basic assumption

Let \(\Omega\) be a bounded open subset of \(IR^N, N \geq 2,\) let \(p(.) \in C_c(\Omega),\) and \(\nu\) a function weight in \(\Omega\) such that:
\[\nu \in L^1_{loc}(\Omega)\]
\[\nu^{\frac{1}{p(x)-1}} \in L^1_{loc}(\Omega)\]
and
\[\nu^{s(x)} \in L^1_{loc}(\Omega) \text{ where } s(x) \in \left[\frac{N}{p(x)}, \infty\right] \cap \left[\frac{1}{p(x)-1}, \infty\right].\]
Let A Leray-Lions operator defined from \(W^{1,p(x)}_0(\Omega, \nu)\) to its dual \(W^{-1,p'(x)}(\Omega, \nu^*)\) by
\[Au = -\text{div} a(x, u, \nabla u),\]
where \(a\) is a Carathéodory function satisfying the following assumptions:
\[|a_i(x, r, \zeta)| \leq \beta \nu^{\frac{p_i}{p(x)}} \left|b(x) + |r|^{\frac{p_i}{p(x)}} + \nu^{\frac{1}{p(x)}}|\zeta|^{p(x)-1}\right|, i = 1, \ldots, N.\]
\[a(x, r, \zeta) - a(x, r, \zeta) \geq \alpha |\zeta|, \quad \zeta \in IR^N.\]
\[a(x, s, \zeta) \zeta \geq \alpha s|\zeta|^{p(x)}\]
with \(b(x)\) be a positive function in \(L^{p(x)}(\Omega),\) and \(\alpha, \beta\) are two strictly positive constants. On another side, let \(g: \Omega \times IR \times IR^N \longrightarrow IR\) a Carathéodory function which satisfied a.e. for \(x \in \Omega\) and for all \(r \in IR, \zeta \in IR^N\) the following conditions:
\[g(x, r, \zeta) r \geq 0.\]
with \(d: IR^+ \longrightarrow IR^+\) a continuous, nondecreasing and positive function; whereas \(c(x)\) be a positive function in \(L^1(\Omega).\)
Moreover, we suppose that \(g\) checks:
\[
\begin{cases}
\exists \rho_1 > 0, \exists \rho_2 > 0 \text{ such that } : \\
\text{for } |s| \geq \rho_1, |g(x, s, \xi)| \geq \rho_2 |\xi|^{p(x)}
\end{cases}
\tag{11}\]
and
\[f \in L^1(\Omega).\]

We have the following theorem

**Theorem 1** Assume that \((3) - (12)) holds, then the problem \((1)\) admits at least one solution \(u \in W^{1,p(x)}_0(\Omega, \nu)\)

**Remark 2** If \(f \in W^{-1,p'(x)}(\Omega, \nu^*)\) then we have \(ug(x, u, \nabla u) \in L^1(\Omega).\)

**Counter example:**
Let us consider \(\Omega = \{x \in IR^2 : |x| < 1\},\) the ball open unit of \(IR^2,\) let \(\lambda \in \left[\frac{1}{2}, 1\right],\) we then have \(u(x) = ln^{\lambda}(\frac{1}{1-|x|}) \in H^1_0(\Omega)\) and \(\Delta u \in L^1(\Omega),\) such that \(-\Delta u \geq 0, u|\nabla u|^2 \in L^1(\Omega)\) and \(|u|^2|\nabla u|^2\) does not belong to \(L^1(\Omega)\) see [7].

**Remark 3** The result of theorem \([1]\) is not true when \(g(x, r, \zeta) = 0,\) seeing that in the non-degenerate case with \(p(x) = p\) constant, \(p \leq N,\) and when \(f \in L^1(\Omega),\) a solution of \(Au = f\) does not belong to \(W^{1,p}(\Omega)\) but belongs to \(\bigcup_{1<q<N} W^{1,q}(\Omega),\) see [8].

**Definition 2** Let \(X\) a Banach space. An operator \(A\) from \(X\) to its dual \(X^*\) is called as type \((M)\) if for all sequence \((u_n)_n \subset X\) satisfying:
\[
\begin{align*}
(i) & \ u_n \rightharpoonup u \text{ weakly in } X \\
(ii) & \ Au_n \rightharpoonup \chi \text{ weakly in } X^* \\
(iii) & \limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \chi, u \rangle,
\end{align*}
\]
then \(\chi = Au.\)

**Remark 4** The theorem 2.1 (p.171 [15]) remains valid if we replace \(A\) Monotous by: \(A\) of type \((M)\).

4 Approximate problem

Let \((f_n)_n\) a sequence of regularly functions such that \(f_n \rightharpoonup f\) in \(L^1(\Omega)\) and \(\|f_n\| \leq \|f\| = C_1.\)
Let us consider the following approximate problem:
\[
(P_n)
\begin{cases}
Au_n + g_n(x, u_n, \nabla u_n) = f_n \text{ in } \Omega, \\
u_n \in W^{1,p(x)}_0(\Omega, \nu),
\end{cases}
\tag{P_n}\]
where \(g_n(x, r, \zeta) = \frac{g(x, r, \zeta)}{1 + |g(x, r, \zeta)|^\alpha} \chi_{\Omega_n},\) with \(\chi_{\Omega_n}\) is the characteristic function of \(\Omega_n\) where \(\Omega_n\) is a sequence of compact subsets which is increasing towards \(\Omega.\)
We have \( g_n(x, r, \zeta) = 0 \) if \( |g_n(x, r, \zeta)| \leq |g(x, r, \zeta)| \) and \( |g_n(x, r, \zeta)| \leq n \).

Let us consider \( G_n : W^{1,p(x)}(\Omega, v) \rightarrow W^{-1,p'}(\Omega, v^*) \) defined by

\[
\langle G_n u, v \rangle = \int_{\Omega} g_n(x, u, \nabla u) v dx, \quad u, v \in W^{1,p(x)}(\Omega, v).
\]

**Proposition 6** The operator \( G_n \) is bounded.

**Proof:**

Let \( u \) and \( v \) in \( W^{1,p(x)}(\Omega, v) \), according to the Hölder inequality, we have:

\[
\langle G_n u, v \rangle = \int_{\Omega} g_n(x, u, \nabla u) v dx \leq \int_{\Omega} |g_n(x, u, \nabla u)| |v(x)|^{\frac{p}{p-1}} dx \leq \left( \frac{1}{p} \right) \left\| g_n(x, u, \nabla u) \right\|_{L^p(\Omega, v)} \left\| v(x) \right\|_{L^{p-1}(\Omega, v)} dx
\]

\[
\langle G_n u, v \rangle \leq C \left\| g_n(x, u, \nabla u) \right\|_{L^p(\Omega, v)} \left\| v(x) \right\|_{W^{1,p(x)}(\Omega, v)} \leq C \left\| v \right\|_{W^{1,p(x)}(\Omega, v)}
\]

On another side \( g_n(x, u + tv, V(u + tv)) \rightarrow g_n(x, u + t_0v, V(u + t_0v)) \), when \( t \rightarrow t_0 \) a.e. \( x \in \Omega \), moreover

\[
\int_{\Omega} |g_n(x, u + tv, V(u + tv))|^{p'(x)} |dx| \leq \left( \frac{1}{n} \right)^{p'-1} |\Omega| < \infty,
\]

while using, still the lemma \( \ref{} \) we obtain:

\[
g_n(x, u + tv, V(u + tv)) \rightarrow g_n(x, u + t_0v, V(u + t_0v))
\]

weakly in \( L^{p'(x)}(\Omega) \), when \( t \rightarrow t_0 \).

Seeing that \( w \in L^{p'(x)}(\Omega) \), we will have:

\[
\langle G_n(u + tv), w \rangle \rightarrow \langle G_n(u + t_0v), w \rangle, \quad \text{when} \ t \rightarrow t_0.
\]

Now, we will show that \( A + G_n \) satisfies the property of type \( (M) \), i.e. that, for all sequence \( (u_j) \subset W^{1,p(x)}(\Omega, v) \) checking:

\[
\begin{align*}
(1) & \quad u_j \rightarrow u \text{ in } W^{1,p(x)}(\Omega, v) \\
(2) & \quad (A + G_n)u_j \rightharpoonup \chi \text{ weakly in } W^{-1,p'(x)}(\Omega, v^*) \\
(3) & \quad \limsup_{j \rightarrow \infty} \| (A + G_n)u_j, u_j - u \|_{p(x)} \leq 0,
\end{align*}
\]

Then \( \chi = (A + G_n)u \).

Indeed:

\[
\int_{\Omega} g_n(x, u_j, \nabla u_j)(u_j - u) dx \leq \| g_n(x, u_j, \nabla u_j) \|_{L^p(\Omega, v^*)} \| u_j - u \|_{L^{p(x)}}
\]

\[
\leq C\left( \int_{\Omega} |g_n(x, u_j, \nabla u_j)|^{p'(x)} dx \right)^{\frac{1}{p'}} \| u_j - u \|_{L^{p(x)}} \quad \text{Thus}
\]

\[
\lim_{j \rightarrow \infty} \langle G_n u_j, u_j - u \rangle = 0.
\]

Consequently, according to \( (iii) \), we obtain

\[
\lim_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle \leq 0,
\]

and seeing that \( A \) is pseudo-monotonic \( \ref{} \), we deduce that

\[
Au_j \rightarrow Au \quad \text{weakly in } W^{-1,p'(x)}(\Omega, v^*)
\]

and \( \lim_{j \rightarrow \infty} \langle Au_j, u_j - u \rangle = 0 \).

On another side

\[
0 = \lim_{j \rightarrow \infty} \int_{\Omega} a(x, u_j, \nabla u_j)(u_j - u) dx
\]

\[
= \lim_{j \rightarrow \infty} \int_{\Omega} \left( \int_{\Omega} a(x, u_j, \nabla u_j) - a(x, u_j, \nabla u) \right)(u_j - u) dx
\]

\[
+ \int_{\Omega} (a(x, u_j, \nabla u)(u_j - u) dx).
\]

Moreover we have \( a(x, u_j, \nabla u) \rightarrow a(x, u, \nabla u) \) strongly in \( L^{p(x)}(\Omega, v^*)^N \), then

\[
\lim_{j \rightarrow \infty} \int_{\Omega} (a(x, u_j, \nabla u) - a(x, u_j, \nabla u))(u_j - u) dx = 0.
\]

Thus

\[
\lim_{j \rightarrow \infty} \int_{\Omega} \left( a(x, u_j, \nabla u) - a(x, u_j, \nabla u) \right)(u_j - u) dx = 0.
\]
Using the lemma see below, we conclude that
\[ \nabla u_j \rightharpoonup \nabla u \text{ a.e. in } \Omega, \text{ when } j \to \infty. \]
Consequently
\[ g_n(x, u_j, \nabla u_j) \rightharpoonup g_n(x, u, \nabla u) \text{ a.e. in } \Omega, \text{ when } j \to \infty. \]

Seeing that \[ |g_n(x, u_j, \nabla u_j)| \leq n \in L^p(\Omega), \]
then according to the dominated convergence theorem of Lebesgue, we obtain:
\[ g_n(x, u_j, \nabla u_j) \to g_n(x, u, \nabla u) \text{ a.e. in } \Omega. \]

Finally
\[ A u_j + G_n u_j \to A u + G_n u = \chi \text{ in } W^{-1,p'(\Omega)}(\Omega, \nabla') \text{ when } j \to \infty. \]

5 Technical lemmas

Lemma 2 Let \( \gamma \) a function weight in \( \Omega \), \( g \in L^{1}(\Omega) \) and \( (g_n)_n \subset L^{1}(\Omega) \) such that \( \|g_n\|_{L^{1}(\Omega)} \leq C \).

If \( g_n \to g \) a.e. in \( \Omega \) then \( g_n \to g \) weakly in \( L^{1}(\Omega) \).

Proof:
Let \( n_0 \geq 1 \), let us pose
\[ E(n_0) = \left\{ x \in \Omega : |g_n(x) - g(x)| \leq 1, \forall n \geq n_0 \right\}. \]
We have
\[ \text{mes}(E(n_0)) \to \text{mes}(\Omega), \text{ when } n_0 \to \infty. \]

Let
\[ F = \left\{ \phi_n \in L^{1}(\Omega) : \phi_n = 0 \text{ a.e. in } \Omega \setminus E(n_0) \right\}. \]

Let us show that \( F \) is dense in \( L^{1}(\Omega) \):

\[ f \in L^{1}(\Omega), \text{ let us pose:} \]
\[ f_n(x) = \begin{cases} f(x) & \text{if } x \in E(n_0), \\ 0 & \text{if } x \in \Omega \setminus E(n_0). \end{cases} \]

We have
\[ \rho_{r'(\Omega)}(f_n(x) - f(x)) = \int_{\Omega} |f_n(x) - f(x)|^{r'(\Omega)} \nu^* dx = \int_{E(n_0)} |f_n(x) - f(x)|^{r'(\Omega)} \nu^* dx + \int_{\Omega \setminus E(n_0)} |f_n(x) - f(x)|^{r'(\Omega)} \nu^* dx, \]

\[ \lim_{n \to \infty} \int_{\Omega} \phi(x)(g_n(x) - g(x)) dx = 0, \forall \phi \in F. \]

According to the dominate convergence theorem, we have
\[ \phi_n \to 0 \text{ a.e. in } \Omega, \]
what implies that
\[ \lim_{n \to \infty} \int_{\Omega} \phi(x)(g_n(x) - g(x)) dx = 0, \forall \phi \in F, \]
and by density of \( F \) in \( L^{1}(\Omega) \), we conclude that:
\[ \lim_{n \to \infty} \int_{\Omega} \phi(x)g_n(x) dx = \int_{\Omega} \phi(x)g(x) dx, \forall \phi \in L^{1}(\Omega), \]
that means
\[ g_n \to g \text{ weakly in } L^{1}(\Omega). \]
Proof: Let us pose

\[ D_n = (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) (\nabla u_n - \nabla u), \]

according the (3.5), we have \( D_n \) is a positive function. We have also \( D_n \to 0 \) in \( L^1(\Omega) \).

Seeing that

\[ u_n \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega, \nu), \]

we have then

\[ u_n \to u \text{ strongly in } L^p(\Omega, \sigma), \]

and consequently

\[ u_n \to u \text{ a.e. in } \Omega. \]

Thanks to \( D_n \to 0 \) a.e. in \( \Omega \), there is then \( B \subset \Omega \) such that \( \text{mes}(B) = 0 \) and for \( x \in \Omega \setminus B \), we have

\[ |u(x)| < \infty, |\nabla u(x)| < \infty, \ u_n(x) \rightharpoonup u(x) \text{ and } D_n(x) \to 0. \]

Let us pose

\[ \xi_n = \nabla u_n(x), \ \xi = \nabla u(x), \]

we have

\[
D_n(x) \geq a \sum_{i=1}^{N} \alpha_i \frac{\omega_i}{\alpha_i |\xi_n|^p} + \alpha \sum_{i=1}^{N} \omega_i |\xi|^p
- \sum_{i=1}^{N} \alpha_i \frac{\omega_i}{\alpha_i |\xi_n|^p} \left[ k(x) + a \frac{p(1)}{p(x)} |u_n|^{\frac{d(x)}{p(x)}} \right],
+ \sum_{j=1}^{N} \omega_j \frac{p(1)}{p(x)} |\xi_j|^p |\xi_n|^{p-1} |\xi_n|,
- \sum_{i=1}^{N} \alpha_i \frac{\omega_i}{\alpha_i |\xi_n|^p} \left[ k(x) + a \frac{p(1)}{p(x)} |u_n|^{\frac{d(x)}{p(x)}} \right],
+ \sum_{j=1}^{N} \omega_j \frac{p(1)}{p(x)} |\xi_j|^p |\xi_n|^{p-1} |\xi_n|,
\geq a \sum_{i=1}^{N} \alpha_i \frac{\omega_i}{\alpha_i |\xi_n|^p} |\xi_n|^p| - C(x)[1 + \sum_{j=1}^{N} \omega_j \frac{1}{p(x)} |\xi_j|^p|^{-1}]
+ \sum_{i=1}^{N} \alpha_i \frac{\omega_i}{\alpha_i |\xi_n|^p} |\xi_n|^{p-1},
\]

where \( C(x) \) is a function depending on \( x \) and not on \( n \). Seeing that \( u_n(x) \rightharpoonup u(x) \), we have \( |u_n(x)| \leq M_x \), where \( M_x \) is positive. Then by a standard argument, we will have \( |\xi_n| \) is uniformly bounded compared to \( n \); indeed:

\[ (4.4) \]

becomes

\[
D_n(x) \geq a \sum_{i=1}^{N} \alpha_i \frac{\omega_i}{\alpha_i |\xi_n|^p} |\xi_n|^p| - C(x) \frac{\omega_i}{N |\xi_n|^p} \frac{1}{|\xi_n|^{p-1}} - C(x) \frac{\omega_i}{|\xi_n|^p} \frac{1}{|\xi_n|^{p-1}}.
\]

If \( |\xi_n| \to \infty \), there is at least \( i_0 \) such that \( |\xi_n^{i_0}| \to \infty \), what will give us \( D_n(x) \to \infty \); what is absurd.

Let \( \xi^* \) an adherent point of \( \xi_n \), we have \( |\xi^*| \to \infty \) and by continuity of the function \( a \) compared to two last variables, we will have:

\[
(a(x, u, \xi^*) - a(x, u, \xi))(\xi^* - \xi) = 0,
\]

and according to (3.2), we obtain \( \xi^* = \xi \), the unicity of the adherent point implies \( \nabla u_n(x) \to \nabla u(x) \) a.e. in \( \Omega \).

Seeing that the sequence \( (a(x, u_n, \nabla u_n))_n \) is bounded in \( (L^{p(x)}(\Omega, \nu)) \), and \( a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \) a.e. in \( \Omega \), then according the lemma 2, we obtain

\[
a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p(x)}(\Omega, \nu)).
\]

let us pose \( \tilde{\nu}_n = a(x, u_n, \nabla u_n) \rightharpoonup \tilde{\nu}_n \) and \( \tilde{\nu} = a(x, u, \nabla u) \rightharpoonup \tilde{\nu} \), in the same way that in [4], we can write

\[
\tilde{\nu}_n = a(x, u_n, \nabla u_n) \rightharpoonup \tilde{\nu} = a(x, u, \nabla u) \rightharpoonup \tilde{\nu} \text{ strongly in } L^1(\Omega).
\]

According to (3.3), we have

\[
\sum_{i=1}^{N} \frac{1}{\alpha_i} \frac{\partial u_n}{\partial x_i} p(x) \leq \tilde{\nu}_n.
\]

let

\[
z_n = \sum_{i=1}^{N} \frac{1}{\alpha_i} \frac{\partial u_n}{\partial x_i} p(x), \ z = \sum_{i=1}^{N} \frac{1}{\alpha_i} \frac{\partial u}{\partial x_i} p(x), \ y_n = \tilde{\nu}_n \frac{1}{\alpha}, \ y = \tilde{\nu} \frac{1}{\alpha}.
\]

Thanks to Fatou lemma, we obtain

\[
\int_{\Omega} 2y dx \leq \liminf_{n \to \infty} \int_{\Omega} y + y_n - z_n - z |dx,
\]

ie

\[
0 \leq - \limsup_{n \to \infty} \int_{\Omega} z_n - z |dx,
\]

then

\[
0 \leq \liminf_{n \to \infty} \int_{\Omega} z_n - z |dx \leq \limsup_{n \to \infty} \int_{\Omega} z_n - z |dx \leq 0,
\]

what implies

\[ \nabla u_n \rightharpoonup \nabla u \text{ in } (L^{p(x)}(\Omega, \nu))^N, \]

consequently

\[ u_n \to u \text{ in } W_0^{1,p(x)}(\Omega, \nu). \]

Definition 3 for any \( k > 0 \) and \( s \in \mathbb{R} \), the truncature function \( T_k(s) \) is defined as:

\[
T_k(s) = \begin{cases} 
    s & \text{if } |s| \leq k, \\
    \frac{k}{|s|} & \text{if } |s| > k.
\end{cases}
\]

Lemma 4 Let \( (u_n)_n \) a sequence from \( W_0^{1,p(x)}(\Omega, \nu) \) such that \( u_n \rightharpoonup u \) weakly in \( W_0^{1,p(x)}(\Omega, \nu) \). Then \( T_k(u_n) \to T_k(u) \) weakly in \( W_0^{1,p(x)}(\Omega, \nu) \).
Proof
We have $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega, \nu)$, and $W_0^{1,p(x)}(\Omega, \nu) \hookrightarrow L^p(\Omega)$, we will have $u_n \to u$ strongly in $L^p(\Omega)$ and a.e. in $\Omega$, consequently $T_k(u_n) \rightharpoonup T_k(u)$ a.e. in $\Omega$.

On another side
\[
\|T_k(u_n)\|_{L^p(\Omega)} \leq \int_\Omega |\nabla T_k(u_n)|^{p(x)} v(x) dx,
\]
\[
\leq \int_\Omega |\nabla u_n|^{p(x)} v(x) dx,
\]
\[
\leq \|u_n\|_{L^p(\Omega)}^{\theta_1} v(x) dx,
\]
where
\[
\theta_1 = \begin{cases} 
  p^+ & \text{if } \|T_k(u_n)\|_{L^p(\Omega)} \leq 1,
  
  p^- & \text{if } \|T_k(u_n)\|_{L^p(\Omega)} > 1,
\end{cases}
\]
and
\[
\gamma = \begin{cases} 
  p^+ & \text{if } \|\nabla u_n\|_{L^p(\Omega)} \leq 1,
  
  p^- & \text{if } \|\nabla u_n\|_{L^p(\Omega)} < 1.
\end{cases}
\]

Thus $(T_k(u_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega, \nu)$, consequently $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p(x)}(\Omega, \nu)$.

6 Proof of theorem

Step1: a priori estimate
The problem $(P_n)$ admits at least a solution $u_n$ belonging to $W_0^{1,p(x)}(\Omega, \nu)$. Choosing $T_k(u_n)$ as test function in $(P_n)$, and seeing that
\[
\int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n) dx \geq 0,
\]
we obtain:
\[
\int_\Omega a(x, u_n, \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_\Omega f_n T_k(u_n) dx.
\]
Using (5), we deduce that
\[
\alpha \int_\Omega v(x) |\nabla T_k(u_n)|^{p(x)} dx \leq k C_1,
\]
that means
\[
\int_\Omega v(x) |\nabla T_k(u_n)|^{p(x)} dx \leq \frac{k}{\alpha} C_1.
\]
Consequently
\[
|\nabla T_k(u_n)|_{L^p(\Omega)}^{\gamma} \leq C_2,
\]
where
\[
\gamma = \begin{cases} 
  p^+ & \text{if } |\nabla T_k(u_n)|_{L^p(\Omega)} \leq 1,
  
  p^- & \text{if } |\nabla T_k(u_n)|_{L^p(\Omega)} < 1.
\end{cases}
\]
On the other hand, we have
\[
\int_\Omega a(x, u_n, \nabla T_k(u_n)) \nabla T_k(u_n) dx
\]
\[
+ \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n) dx = \int_\Omega f_n T_k(u_n) dx.
\]
What implies
\[
\int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n) dx \leq \int_\Omega |f_n| dx.
\]
Thus
\[
\int_{|u_n| > k} k |u_n| g_n(x, u_n, \nabla u_n) dx
\]
\[
+ \int_{|u_n| \leq k} g_n(x, u_n, \nabla u_n) u_n dx \leq \int_\Omega |f_n| dx.
\]
Consequently
\[
k \int_{|u_n| > k} g_n(x, u_n, \nabla u_n) dx \leq k C_1
\]
However
\[
\int_\Omega v(x) |\nabla u_n|^{p(x)} dx \leq \int_{|u_n| > k} v(x) |\nabla u_n|^{p(x)} dx
\]
\[
+ \int_\Omega v(x) |\nabla u_n|^{p(x)} dx.
\]
Then for $k > p_1$, we will have:
\[
\int_\Omega v(x) |\nabla u_n|^{p(x)} dx \leq \frac{1}{p^2_2} \int_{|u_n| > k} g_n(x, u_n, \nabla u_n) dx + C_2 \leq C_4
\]
What implies that a sequence $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega, \nu)$.

Step2: Strong convergence of truncations
Seeing that $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega, \nu)$, and $W_0^{1,p(x)}(\Omega, \nu) \hookrightarrow L^p(\Omega)$, we can extract a subsequence of $(u_n)_n$, still noted $(u_n)_n$, and there is $u \in W_0^{1,p(x)}(\Omega, \nu)$ such that
\[
\left\{ \begin{array}{l}
  u_n \to u \quad \text{in } W_0^{1,p(x)}(\Omega, \nu), \\
  u_n \to u \quad \text{a.e. in } \Omega,
\end{array} \right.
\]
We want to show that $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p(x)}(\Omega, \nu)$.
Let $z_n = T_k(u_n) - T_k(u)$, let us pose $v_n = \varphi_A(z_n)$, where $\varphi_A(s) = se^{s^2}$.
Choosing $v_n$ as test function in $(P_n)$.
We are $(v_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega, \nu)$, and $v_n \to 0$ a.e. in $\Omega$, then according to lemma 4 we obtain $v_n \to 0$ weakly in $W_0^{1,p(x)}(\Omega, \nu)$.
thus $\langle f_n, v_n \rangle \to 0$, because $v_n \to 0$ weakly in $L^\infty(\Omega)$, and $f_n \to f$ strongly in $L^1(\Omega)$.
Consequently
\[
\eta_{1n} = \langle Au_n, v_n \rangle + \langle G_n u_n, v_n \rangle \to 0.
\]
 Seeing that $g_n(x, u_n, \nabla u_n) \geq 0$ on $\{x \in \Omega : |u_n| \geq k\}$, then we have:
\[
\langle Au_n, v_n \rangle + \int_{|u_n| \geq k} g_n(x, u_n, \nabla u_n) v_n dx \leq \eta_{1n}.
\]
Thus
\[
\langle Au_n, v_n \rangle - \int_{|u_n| \leq k} g_n(x, u_n, \nabla u_n) v_n dx \leq \eta_{1n} - \eta_{1n} (14).
\]
however
\[
\langle A u_n, v_n \rangle = \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (T_k(u_n) - T_k(u)) \varphi_\lambda'(z_n) dx \leq 2 (\eta_{1n} + \eta_{2n} + \eta_{4n}) \rightarrow 0 \quad \text{when } n \rightarrow \infty.
\]

Thanks to lemma 3, we deduce that
\[
T_k(u_n) \rightarrow T_k(u) \quad \text{strongly in } W_0^{1,p(x)}(\Omega, v).
\]

step3: Equi-integrability of the nonlinearities \((g_n(x, u_n, \nabla u_n))_n\).

We will show that \(g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u)\) strongly in \(L^1(\Omega)\). Seeing (17), we deduce that
\[
\nabla u_n \rightarrow \nabla u \quad \text{a.e. in } \Omega, \quad \text{when } n \rightarrow \infty.
\]

Let \(E\) a measurable part of \(\Omega\), and let \(m > 0\), let us pose
\[
X_m^0 = \{ x \in \Omega : |u_n| \leq m \} \quad \text{et} \quad (X_m^0)^C = \{ x \in \Omega : |u_n| > m \}.
\]

We have
\[
\int_E |g_n(x, u_n, \nabla u_n)| dx + \int_{E^{\tilde{\Omega}}(X_m^0)^C} |g_n(x, u_n, \nabla u_n)| dx \leq d(m) \int_E |v(x)||V T_m(u_n)|^{p(x)} + c(x) dx + \int_{(X_m^0)^C} |g_n(x, u_n, \nabla u_n)| dx.
\]

Thanks to (17) and seeing \(c(x) \in L^1(\Omega)\), there is \(\delta > 0\) such that for all \(E: |E| < \delta\), then
\[
d(m) \int_E (v(x)||V T_m(u_n)|^{p(x)} + c(x) dx \leq \frac{\eta}{2}.
\]

Let \(\psi_m\) a function defined by:
\[
\psi_m(x) = \begin{cases} 0 & \text{if } |x| \leq m - 1, \\ 1 & \text{if } s \geq m, \\ 1 & \text{if } s \leq m - m, \end{cases}
\]

we have \(\psi_m(u_n) \in W_0^{1,p(x)}(\Omega, v)\), choosing \(\psi_m(u_n)\) as test function in (P_n), we obtain:
\[
\int a(x, u_n, \nabla u_n) \nabla \psi_m(u_n) dx + \int g_n(x, u_n, \nabla u_n) \psi_m(u_n) dx = \int f_n \psi_m(u_n) dx.
\]

According (9) and (10), we deduce that
\[
\int_{[|u_n| > m - 1]} |g_n(x, u_n, \nabla u_n)| dx \leq \int_{[|u_n| > m - 1]} |f_n| dx.
\]

what implies
\[
\int_{[|u_n| > m]} |g_n(x, u_n, \nabla u_n)| dx \leq \int_{[|u_n| > m - 1]} |f_n| dx.
\]

however \(f_n \rightarrow f \) in \(L^1(\Omega)\) and \([|u_n| > m - 1] \rightarrow 0\) when \(m \rightarrow \infty\), uniformly in \(n\), then for \(m\) big enough we have;
\[
\int_{[|u_n| > m - 1]} |f_n| dx \leq \frac{\eta}{2}, \quad \forall n \in \mathbb{N}.
\]
Thus
\[
\int_{|u_n| \geq m} |g_n(x,u_n,\nabla u_n)| dx \leq \frac{\eta}{2}, \quad \forall n \in \mathbb{N}.
\] (21)

Combining (19), (20) and (21), we deduce that
\[
\int_{E} |g_n(x,u_n,\nabla u_n)| dx \leq \eta, \quad \text{for all } n \in \mathbb{N}.
\]

That means that the sequence \( (g_n(x,u_n,\nabla u_n)) \) is equi-integrable, but \( g_n(x,u_n,\nabla u_n) \rightarrow g(x,u,\nabla u) \) a.e. in \( \Omega \), then thanks to Vitali theorem, we deduce that
\[
g_n(x,u_n,\nabla u_n) \rightarrow g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega).
\]

step 4: passing to the limit

Seeing that \( (u_n) \) is bounded in \( W^{1,p(x)}_{0}(\Omega, v) \), and thanks to (6), we conclude that \( (a(x,u_n,\nabla u_n)) \) is bounded in \( (L^p(x)(\Omega, v^*) \right)^N \), and seeing that \( a(x,u_n,\nabla u_n) \rightarrow a(x,u,\nabla u) \) a.e. in \( \Omega \), then according to the lemma 2 we obtain that
\[
a(x,u_n,\nabla u_n) \rightarrow a(x,u,\nabla u) \quad \text{weakly in } (L^{p^*}(\Omega, v^*) \right)^N,
\]
while making \( n \rightarrow \infty \), we obtain \( \forall v \in W^{1,p(x)}_{0}(\Omega, v) \cap L^\infty(\Omega),
\]
\[
\langle Au, v \rangle + \int_{\Omega} g(x,u,\nabla u)v dx = \int_{\Omega} f v dx.
\]

7 Example of application

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N, \) \( N \geq 2 \), and let \( p(x), q(x) \in C_0(\overline{\Omega}) \).

Let us pose:
\[
\begin{align*}
& a_i(x,r,\zeta) = \nu(x)(\zeta)^{p(x)-1} s g_n(\zeta), \quad i = 1, ..., N, \\
& g(x,r,\zeta) = \rho r |\nu(\zeta)|^{p(x)} |\zeta|^{p(x)}, \quad \rho > 0,
\end{align*}
\]

were \( \nu(x) \) a function weight in \( \Omega \). The function \( a_i, \)
\( i = 1, ..., N, \) which satisfies the assumptions of theorem 9, 10 and 11, with \( |s| \geq \rho_1 = 1 \) and \( \rho_2 = \rho > 0, \)

consequently the assumptions of theorem 1 are satisfied; thus for \( f \in L^1(\Omega), \) the following theorem, with \( \nu(x) = d^{x}(x): \)

\[
\begin{align*}
\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( d^{x}(x) \frac{\partial u}{\partial x_i} |\nu(\zeta)|^{p(x)} |\zeta|^{p(x)} \right)
\end{align*}
\]

admits at least one solution \( u \in W^{1,p(x)}_{0}(\Omega, d^{x}(x)). \)

References