Existence Results for Nonlinear Anisotropic Elliptic Equation

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\begin{abstract}
In this work, we shall be concerned with the existence of weak solutions of anisotropic elliptic operators \(Au + \sum_{i=1}^{N} g_i(x,u,Vu) + \sum_{i=1}^{N} H_i(x,u,Vu) = f - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i\), where the right hand side \(f\) belongs to \(L^{2^*}(\Omega)\) and \(k_i\) belongs to \(L^{p_i}(\Omega)\) for \(i = 1,\ldots, N\) and \(A\) is a Leray-Lions operator. The critical growth condition on \(g_i\) is the respect to \(u\) and no growth condition with respect to \(u\), while the function \(H_i\) grows as \(|\nabla u|^{p_i-1}\).
\end{abstract}

1 Introduction

In this paper we study the existence of weak solutions to anisotropic elliptic equations with homogeneous Dirichlet boundary conditions of the type

\[
\begin{aligned}
Au + \sum_{i=1}^{N} g_i(x,u,Vu) + \sum_{i=1}^{N} H_i(x,u,Vu) &= f - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i, \\
    u &= 0
\end{aligned}
\]

in \(\Omega\), where \(\Omega\) is a bounded open subset of \(\mathbb{R}^N (N \geq 2)\) with Lipschitz continuous boundary. The operator \(Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u,Vu)\) is a Leray-Lions operator such that the functions \(a_i\), \(g_i\) and \(H_i\) are Carathodory functions satisfying the following conditions for all \(s \in \mathbb{R}, \xi \in \mathbb{R}^N, \xi' \in \mathbb{R}^N\) and a.e. in \(\Omega\):

\[
\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \lambda \sum_{i=1}^{N} |\xi_i|^{p_i},
\]

\[
|a_i(x,s,\xi)| \leq \gamma |s|^{p_i} + |\xi_i|^{p_i-1},
\]

\[
(g_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i',
\]

\[
g_i(x,s,\xi)s \geq 0,
\]

\[
|g_i(x,s,\xi)| \leq L(|s|)|\xi_i|^{p_i} \quad \forall i = 1,\ldots, N,
\]

where \(\lambda, \gamma, h_1\) are some positive constants, for \(i = 1,\ldots, N\) and \(L : \mathbb{R}^+ \to \mathbb{R}^+\) is a continuous and non decreasing function. The right hand side \(f\) and \(k_i\) for \(i = 1,\ldots, N\) are functions belonging to \(L^{2^*}(\Omega)\) and \(L^{p_i}(\Omega)\) where \(p_i^* = \frac{p_i}{p_i - 1}, p_{\infty} = \frac{p_{\infty}}{p_{\infty} - 1}\), with \(p_{\infty} = \max\{p_1,\ldots, p_N\}\), where \(p = \max\{p_1,\ldots, p_N\}\), \(\overline{p} = \frac{1}{\sum_{i=1}^{N} \frac{1}{p_i}}\) and \(\overline{p}_\infty = \frac{N\overline{p}}{N-\overline{p}}\).

Since the growth and the coercivity conditions of each \(a_i\) for all \(i = 1,\ldots, N\) depend on \(p_i\), we need to use the anisotropic Sobolev space. We mention some papers on anisotropic Sobolev spaces (see e.g. [1]-[5]).

If \(p_i = p\) for all \(i = 1,\ldots, N\), we refer some works such as by Guibé in [6], by Monetti and Randazzo in [7] and by Y. Akdim, A. Benkirane and M. El Moumen in [8].

In [3], L.Boccardo, T. Gallouet and P. Marcellini have studied the problem [1] when \(a_i(x,u,Vu) = \left|\frac{\partial}{\partial x_i}\right|^{p_i-2}\frac{\partial}{\partial x_i} g_i = 0, H_i = 0, k_i = 0 \) and \(f = 0\) is Radon's measure. In [3], F. Li has proved the existence and regularity of weak solutions of the problem [1] with \(g_i = 0, H_i = 0, k_i = 0 \) for all \(i = 1,\ldots, N\) and \(f\) belongs to \(L^{2^*}(\Omega)\) with \(m > 1\). In [9], R. Di Nardo and F. Feo have proved the existence of weak solution of the problem [1] when \(g_i = 0, k_i = 0, H_i > 0 \) for all \(i = 1,\ldots, N\).

In this work, we prove the existence of weak solu-
2 Preliminaries

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N (N \geq 2) \) with Lipschitz continuous boundary and let \( 1 < p_1, \ldots, p_N < \infty \) be \( N \) real numbers, \( p^* = \max \{p_1, \ldots, p_N\} \), and \( \overline{p} = (p_1, \ldots, p_N) \). The anisotropic Sobolev space (see [12])

\[
W^{1,\overline{p}}(\Omega) = \left\{ u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), i = 1, 2, \ldots, N \right\}
\]

is a Banach space with respect to norm

\[
\|u\|_{W^{1,\overline{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^N \|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)}.
\]

The space \( W^{1,\overline{p}}_0(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) with respect to this norm. We recall a Poincaré-type inequality.

Let \( u \in W^{1,\overline{p}}_0(\Omega) \) then there exists a constant \( C_p \) such that (see [13])

\[
\|u\|_{L^p(\Omega)} \leq C_p \|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)} \text{ for } i = 1, \ldots, N.
\]

Moreover a Sobolev-type inequality holds. Let us denote by \( \overline{p} \) the harmonic mean of these numbers, i.e.,

\[
\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}.
\]

Let \( u \in W^{1,\overline{p}}_0(\Omega) \), then there exists (see [12]) a constant \( C_s \) such that

\[
\|u\|_{L^q(\Omega)} \leq C_s \left( \sum_{i=1}^N \left( \frac{\partial u}{\partial x_i} \right)^{1/p_i} \right)^{1/q} \|u\|_{L^{p_i}(\Omega)}.
\]

Where \( q = \frac{Nq}{\overline{p}} \) if \( \overline{p} < N \) or \( q \in [1, +\infty[ \) if \( \overline{p} \geq N \). We recall the arithmetic mean: Let \( a_1, \ldots, a_N \) be positive numbers, it holds

\[
\frac{1}{N} \sum_{i=1}^N a_i \leq \left( \frac{1}{N} \sum_{i=1}^N a_i^\alpha \right)^{1/\alpha} \leq \frac{1}{N} \sum_{i=1}^N a_i.
\]

Which implies by \([2]\)

\[
\|u\|_{L^q(\Omega)} \leq C_s \frac{1}{N} \sum_{i=1}^N \left( \frac{\partial u}{\partial x_i} \right)^{1/p_i} \|u\|_{L^{p_i}(\Omega)}.
\]

When \( \overline{p} < N \) hold, inequality \([3]\) implies the continuous embedding of the space \( W^{1,\overline{p}}_0(\Omega) \) into \( L^q(\Omega) \) for every \( q \in [1, \overline{p}] \). On the other hand the continuity of the embedding \( W^{1,\overline{p}}_0(\Omega) \hookrightarrow L^q(\Omega) \) relies on inequality \([1]\). Let us put \( p_\infty := \max\{p_\star, p^*\} \)

**Proposition 1** For \( q \in [1, p_\infty] \) there is a continuous embedding \( W^{1,\overline{p}}_0(\Omega) \hookrightarrow L^q(\Omega) \). If \( q < p_\infty \) the embedding is compact.

### 3 Assumptions and Definition

We consider the following class of nonlinear anisotropic elliptic homogenous Dirichlet problems

\[
\begin{align*}
-\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + \sum_{i=1}^N g_i(x, u, \nabla u) + & \\
\sum_{i=1}^N H_i(x, \nabla u) = f - \sum_{i=1}^N \frac{\partial}{\partial x_i} k_i \text{ in } \Omega, \\
\quad u = 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N (N \geq 2) \) with Lipschitz continuous boundary \( \partial \Omega \), \( 1 < p_1, \ldots, p_N < \infty \). We assume that \( a_i : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}, g_i : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R} \) and \( H_i : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R} \) are Carathéodory functions such that for all \( s \in \mathbb{R}, \xi \in \mathbb{R}^N \) and a.e. in \( \Omega \):

\[
\begin{align*}
\sum_{i=1}^N a_i(x, s, \xi) \xi_i \geq \lambda \sum_{i=1}^N |\xi_i|^{p_i}, \\
\sum_{i=1}^N \left| a_i(x, s, \xi) \right| \leq \gamma \left( |s|^{\overline{p}} + |\xi|^{p_i} \right),
\end{align*}
\]

(5)

\[
\begin{align*}
|g_i(x, s, \xi)| \leq |g_i(x, s)| \\
|g_i(x, s, \xi)| \leq |g_i(x, s)| + \lambda \sum_{i=1}^N |\xi_i|^{p_i} \forall i = 1, \ldots, N,
\end{align*}
\]

(6)

\[
\begin{align*}
|H_i(x, \xi)| \leq b_i |\xi|^{p_i-1},
\end{align*}
\]

(10)

where \( \lambda, \gamma, b_i \) are some positive constants, for \( i = 1, \ldots, N \) and \( L : \mathbb{R}^N \rightarrow \mathbb{R}^+ \) is a continuous and non decreasing function. Moreover, we suppose that

\[
\begin{align*}
f & \in L^{p_\star}(\Omega), \\
\quad k_i \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, \ldots, N.
\end{align*}
\]

(11)

### Definition 1

A function \( u \in W^{1,\overline{p}}_0(\Omega) \) is a weak solution of the problem \([1]\) if \( \sum_{i=1}^N g_i(x, u, \nabla u) \in L^1(\Omega) \) and \( u \) satisfies

\[
\begin{align*}
\sum_{i=1}^N \int_{\Omega} \left[ a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + g_i(x, u, \nabla u) \varphi + H_i(x, \nabla u) \varphi \right] \\
= \int_{\Omega} \left[ f \varphi + \sum_{i=1}^N k_i \frac{\partial \varphi}{\partial x_i} \right]
\end{align*}
\]

(12)

\[ \forall \varphi \in W^{1,\overline{p}}_0(\Omega) \cap L^\infty(\Omega). \]
4 Main results

In this section we prove the existence of at least a weak solution of the problem [1]. We consider the approximate problems.

4.1 Approximate problems and a priori estimates

Let

\[ g_i^n(x, u, \nabla u) = \frac{g_i(x, u, \nabla u)}{1 + \frac{1}{n}|g_i(x, u, \nabla u)|} \]

and

\[ H_i^n(x, \nabla u) = \frac{H_i(x, \nabla u)}{1 + \frac{1}{n}|H_i(x, \nabla u)|} \]

By Leray-Lions (see e.g. [14]), there exists at least a weak solution \( u_n \in W_0^{1,p} (\Omega) \) of the following approximate problem

\[
\begin{align*}
\left\{ \begin{array}{l}
- \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u_n, \nabla u_n) + \sum_{i=1}^{N} g_i^n(x, u_n, \nabla u_n) \\
\quad + \sum_{i=1}^{N} H_i^n(x, \nabla u_n) = f - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i \quad \text{in } \Omega \\
\quad \text{on } \partial \Omega.
\end{array} \right.
\]

(13)

**Lemma 1** See ([9], lemma 4.2) Let \( A \in \mathbb{R}^+ \) and \( u \in W_0^{1,p} (\Omega) \), then there exists \( t \) measurable subsets \( \Omega_1, \ldots, \Omega_t \) of \( \Omega \) and \( t \) functions \( u_1, \ldots, u_t \) such that \( \Omega_1 \cup \cdots \cup \Omega_t = \Omega \) and \( |\Omega_1| = A \) for \( s \in \{1, \ldots, 1\} \), \( x \in \Omega : |\frac{\partial u}{\partial x_i}| = 0 \) for \( i = 1, \ldots, t \), \( N \subseteq \Omega_t, \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x_i} \) a.e. in \( \Omega_t \), \( \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x_i} \) \( u \in \Omega \) and \( \text{sign}(u) = \text{sign}(u_n) \) if \( u \notin \Omega \).

**Proposition 2** Assume that \( \bar{p} < N \), ([12] hold and let \( u_n \in W_0^{1,p} (\Omega) \) be a solution to problem (13) then, we have

\[
\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right|^p \leq C,
\]

(14)

for some positive constant \( C \) depending on \( N, \Omega, \lambda, \gamma, p, b, \|f\|_{L^{p'}(\Omega)}, \|g\|_{L^{p'}(\Omega)} \) for \( i = 1, \ldots, N \).

**Proof:** Let \( A \) be a positive real number, that will be chosen later, Refering to lemma 1. Let us fix \( s \in \{1, \ldots, t\} \) and let us use \( T_k(u_n) \) as test function in problem (13) using ([5], [6]), Young’s and Hölder’s inequalities and proposition 1 we obtain

\[
\sum_{i=1}^{N} \int_{|u_n| \leq k} \left| \frac{\partial u_n}{\partial x_i} \right|^p \leq C_1 \left( \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_1} \right)^{\frac{1}{p_1}} + \sum_{k=1}^{N} \int_{|u_n| \leq k} \left| \frac{\partial u_n}{\partial x_i} \right|^p \]

(15)

The dominated convergence theorem implies that

\[
\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right|^p \leq C_1 \left( \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_1} \right)^{\frac{1}{p_1}} + \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right|^p \]

(16)

and in what the constants depend on the data but not on the function \( u \).

Using condition ([10]), Hölder’s and Young’s inequalities, lemma 1 and proposition 1 we get

\[
\sum_{i=1}^{N} \int_{\Omega} \left| H_i(x, \nabla u) \right| |u_n| \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right|^p \]

(17)

for some constant \( C_1 > 0 \), where \( d_i = \prod_{i=1}^{N} \left( \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_1} \right)^{\frac{1}{p_1}} \)

(18)
inequality (17) becomes
\[ \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_i}{\partial x_i} \right|^p \leq C_4 \left( \sum_{i=1}^{N} \int_{\Omega} \frac{\left| f_i \right|}{L^p(\Omega)} \right)^{1/p} \]
for some constant \( C_4 > 0 \) and for \( s = 1 \), we get
\[ \int_{\Omega} \left| \frac{\partial u_i}{\partial x_i} \right|^p \leq \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u_i}{\partial x_i} |p_i| \]
\[ \leq C_4 \left( \sum_{i=1}^{N} \int_{\Omega} \frac{\left| f_i \right|}{L^p(\Omega)} \right)^{1/p} \]
Let us choose \( A \) such that (18) and \( 1 - C_4 \sum_{i=1}^{N} A_i \left| \frac{\partial u_i}{\partial x_i} \right|^p > 0 \) hold, (see [9]).
For example, we can take
\[ A \min \left\{ \left( \frac{1}{x^2} \right)^{\frac{1}{2} \max_{i \leq N \leq 1} |1 - \frac{\partial u_i}{\partial x_i}|}, \left( \frac{1}{x^2} \right)^{\frac{1}{2} \max_{i \leq N \leq 1} |\frac{\partial u_i}{\partial x_i}|} \right\} \]
By this choice, we obtain
\[ d_1 = \min \left\{ \sum_{i=1}^{N} \left( \int_{\Omega} \frac{\partial u_i}{\partial x_i} |p_i| \right)^{1/p} \right\} \]
\[ \leq C_5 \left( \left( \sum_{i=1}^{N} \int_{\Omega} \frac{\left| f_i \right|}{L^p(\Omega)} \right)^{1/p} \left( \sum_{i=1}^{N} \int_{\Omega} \frac{\left| \frac{\partial u_i}{\partial x_i} \right|^p}{L^p(\Omega)} \right)^{1/p} \right) \]
Then there exists a constant \( C_6 > 0 \) such that \( d_1 \leq C_6 \) and by (20), we obtain
\[ \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u_i}{\partial x_i} |p_i| \leq C_7 \]
for some constant \( C_7 > 0 \). Moreover using (21) in (19) and iterating on \( s \), we have
\[ \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_i}{\partial x_i} \right|^p \leq C_9 \]
then arguing as before, we obtain \( \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u_i}{\partial x_i} |p_i| \leq C_9 \),
for some constant \( C_9 > 0 \), then
\[ \left| \int_{\Omega} |u|^p \right|_{L^p(\Omega)} \leq C_{10} \]
for some positive \( k > 0 \).
Since \( u_n \) is bounded in \( L^p(\Omega) \) (the embedding \( L^p(\Omega) \hookrightarrow L^p(\Omega) \)) is compact, we obtain the following results.

**Corollary 1** If \( u_n \) is a weak solution of problem (13), then there exists a subsequence \( (u_{n_k})_k \) such that \( u_n \rightharpoonup u \) weakly in \( L^p(\Omega) \), strongly in \( L^p(\Omega) \) and a.e. in \( \Omega \).

### 4.2 Strong convergence of \( T_k(u_n) \)

**Lemma 2** Assume that \( u_n \rightharpoonup u \) weakly in \( L^p(\Omega) \) and \( a, \epsilon \) in \( \Omega \) and
\[ \sum_{i=1}^{N} \int_{\Omega} \left[ a_i(x, u_n, \nabla u_n) - a_i(x, u, \nabla u) \right] \frac{\partial u_n}{\partial x_i} \cdot \frac{\partial u_n}{\partial x_i} \rightarrow 0 \],
\[ u_n \rightharpoonup u \text{ strongly in } L^p(\Omega) \).

**Proof:** The proof follows as in Lemma 5 of [15] taking into account the anisotropy of operator.

**Proposition 3** Let \( u_n \) be a solution to the approximate problem (13), then
\[ T_k(u_n) \rightarrow T_k(u) \text{ strongly in } L^p(\Omega) \].

**Proof:** Let us fix \( k \) and let \( \delta \) be a real number such that \( \delta \geq \left( \frac{(1/k)}{2} \right)^2 \). Let us define \( z_n = T_k(u_n) - T_k(u) \) and \( \varphi(s) = se^{\delta s^2} \), it is easy to check that for all \( s \in \mathbb{R} \) one has
\[ \varphi'(s) - \frac{L(k)}{\lambda} |\varphi(s)| \geq \frac{1}{2} \]
Using \( \varphi(z_n) \) as test function in (13), we get
\[ \sum_{i=1}^{N} \int_{\Omega} \left[ a_i(x, u_n, \nabla u_n) \frac{\partial \varphi(z_n)}{\partial x_i} \right] + \sum_{i=1}^{N} \int_{\Omega} H_i^\epsilon(x, u_n, \nabla u_n) \varphi(z_n) = \int_{\Omega} f \varphi(z_n) + \sum_{i=1}^{N} \int_{\Omega} \left( \frac{\partial \varphi(z_n)}{\partial x_i} \right) \frac{\partial u_n}{\partial x_i} \]
Since \( \varphi(z_n) \rightarrow 0 \) weakly in \( L^p(\Omega) \) then \( \int_{\Omega} f \varphi(z_n) \rightarrow 0 \) as \( n \rightarrow +\infty \). Since \( T_k(u_n) \rightarrow T_k(u) \) weakly in \( L^p(\Omega) \) and \( \varphi'(z_n) \) is bounded then \( \sum_{i=1}^{N} \int_{\Omega} \left( \frac{\partial \varphi(z_n)}{\partial x_i} \right) \frac{\partial u_n}{\partial x_i} \rightarrow 0 \) as \( n \rightarrow +\infty \). On the other hand, we have
\[ \sum_{i=1}^{N} \int_{\Omega} H_i^\epsilon(x, u_n, \nabla u_n) \varphi(z_n) \]
then dominated convergence theorem \( b_i \varphi(z_n) \rightarrow 0 \) strongly in \( L^p(\Omega) \), then
\[ \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
\[ \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
\[ \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
\[ \leq C \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
Since \( b_i \varphi(z_n) \rightarrow 0 \) a.e. in \( \Omega \) and \( |b_i \varphi(z_n)| \leq |b_i| x 2 \epsilon k^2 \delta \in L^p(\Omega) \), then by dominated convergence theorem \( b_i \varphi(z_n) \rightarrow 0 \) strongly in \( L^p(\Omega) \), then
\[ \sum_{i=1}^{N} \int_{\Omega} H_i^\epsilon(x, u_n, \nabla u_n) \varphi(z_n) \]
\[ \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
\[ \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
\[ \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
\[ \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
\[ \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
\[ \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_n}{\partial x_i} \right| |b_i \varphi(z_n)| \]
On the other hand, we have
\[
\sum_{i=1}^{N} \int_{\Omega} a_{i}(x, u_{n}, \nabla u_{n}) \frac{\partial \varphi(z_{n})}{\partial x_{i}} dx = \sum_{i=1}^{N} \int_{\Omega} a_{i}(x, u_{n}, \nabla u_{n}) \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi'(z_{n}) dx.
\]

Then the sequence \( \left\{ a_{i}(x, u_{n}, \nabla u_{n}) \varphi(z_{n}) \right\} \) is bounded in \( L^{p}(\Omega) \), then by the growth condition \( 6 \), we get
\[
\left| \sum_{i=1}^{N} \int_{\Omega} a_{i}(x, u_{n}, \nabla u_{n}) \frac{\partial \varphi(z_{n})}{\partial x_{i}} dx \right| \leq 2^{p_{i}^{-1}} \left( 1 + 8 \delta k^{2} \right) \int_{\Omega} \left| \sum_{i=1}^{N} \int_{\Omega} a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \frac{\partial T_{k}(u_{n})}{\partial x_{i}} \varphi'(z_{n}) dx \right| \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi(z_{n}) dx.
\]

Then the sequence \( \left\{ a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \varphi(z_{n}) \right\} \) is equi-integrable and by Vitali’s theorem one has \( a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \varphi(z_{n}) \) converges to \( a_{i}(x, T_{k}(u), \nabla T_{k}(u)) \) strongly in \( L^{p}(\Omega) \). Since \( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} \to \frac{\partial T_{k}(u)}{\partial x_{i}} \) weakly in \( L^{p}(\Omega) \), then
\[
\lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u)) \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi'(z_{n}) = 0.
\]

It follows that
\[
\sum_{i=1}^{N} \int_{\Omega} \left( a_{i}(x, u_{n}, \nabla u_{n}) \frac{\partial \varphi(z_{n})}{\partial x_{i}} \right) \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi'(z_{n}) = 0.
\]

On the other hand, by virtue of \( 5 \) and \( 9 \),
\[
\sum_{i=1}^{N} \int_{\Omega} \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi'(z_{n}) + \varepsilon_{5}(n).
\]

Similarly as above, it’s easy to see that by \( 6 \), corollary \( 1 \) and Vitali’s theorem one has
\[
\sum_{i=1}^{N} \int_{\Omega} a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \varphi(z_{n}) dx \leq \sum_{i=1}^{N} \int_{\Omega} a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi(z_{n}) dx.
\]

and taking into account that \( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} \to \frac{\partial T_{k}(u)}{\partial x_{i}} \) weakly in \( L^{p}(\Omega) \), we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi(z_{n}) dx \to 0
\]

where \( \kappa \in L^{p}(\Omega) \). Hence, we get
\[
\sum_{i=1}^{N} \int_{\Omega} \left( a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \right) \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi(z_{n}) dx \to 0
\]

as \( n \to +\infty \). Thanks to \( 6 \) and \( 14 \), the sequence \( \left\{ a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \right\} \) is bounded in \( L^{p}(\Omega) \), so that there exists \( l_{k}^{*} \in L^{p}(\Omega) \) such that \( a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \to l_{k}^{*} \) weakly in \( L^{p}(\Omega) \). We have
\[
\sum_{i=1}^{N} \int_{\Omega} \left( a_{i}(x, T_{k}(u_{n}), \nabla T_{k}(u_{n})) \right) \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi(z_{n}) dx = \sum_{i=1}^{N} \int_{\Omega} l_{k}^{*} \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi(z_{n}) dx.
\]

Since \( \varphi(z_{n}) \to 0 \) weakly in \( L^{\infty}(\Omega) \), we conclude that
\[
\sum_{i=1}^{N} \int_{\Omega} l_{k}^{*} \left( \frac{\partial T_{k}(u_{n})}{\partial x_{i}} - \frac{\partial T_{k}(u)}{\partial x_{i}} \right) \varphi(z_{n}) dx \to 0
\]

as \( n \to +\infty \). Hence, we get
\[ \left| \sum_{i=1}^{N} \int_{[u_n \leq k]} g_i^n(x, u_n, \nabla u_n) \varphi(z_n) dx \right| \leq \frac{L(k)}{\lambda} \sum_{i=1}^{N} \int_{\Omega} A_i(x, T_k(u_n), \nabla T_k(u_n)) \] 

(26)

\[ -a_i(x, T_k(u_n), \nabla T_k(u_n)) \left( \frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i} \right) \varphi(z_n) dx + \epsilon \varepsilon(n). \]

Combining (24), (25) and (26), we obtain

\[ 0 \leq \sum_{i=1}^{N} \int_{\Omega} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u), \nabla T_k(u)) \right) \left( \frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i} \right) \leq 2 \epsilon \varepsilon(n), \]

then Lemma 2 gives

\[ T_k(u_n) \rightarrow T_k(u) \text{ strongly in } W_0^{1, \overline{p}}(\Omega). \] 

(27)

### 4.3 Existence results

**Theorem 1** Assume that \( \overline{p} < N \) and (6)–(12) hold. Then there exists at least a weak solution of the problem (1).

**Proof:** By (4.2) the sequence \( \frac{\partial u_n}{\partial x_i} \) is bounded in \( L^{\overline{p}}(\Omega) \), so that \( \frac{\partial u_n}{\partial x_i} \rightarrow \frac{\partial u}{\partial x_i} \) weakly in \( L^{\overline{p}}(\Omega) \) for \( i = 1, \ldots, N \) and \( u_n \rightarrow u \) strongly in \( L^{\overline{p}}(\Omega) \). By (6) there exists a subsequence, which we still denote by \( u_n \) such that \( \frac{\partial u_n}{\partial x_i} \rightarrow \frac{\partial u}{\partial x_i} \) a. e. in \( \Omega \) for \( i = 1, \ldots, N \), then for \( i = 1, \ldots, N \), we have

\[ \int_{\Omega} A_i(x, u_n, \nabla u_n) \varphi(z_n) dx \leq \frac{L(k)}{\lambda} \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n)) \left( \frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i} \right) \varphi(z_n) dx + \epsilon \varepsilon(n). \]

Moreover by (6) and (10), we have

\[ \int_{\Omega} \left( H_i^n(x, u_n, \nabla u_n) \right) dx \leq C \left( \int_{\Omega} |u_n|^{\overline{p}^*} + \int_{\Omega} |\partial u_n|^{p_i} \right) \] 

(14)

\[ \int_{\Omega} \left( H_i^n(x, u_n, \nabla u_n) \right) dx \leq C \left( \int_{\Omega} |\partial u_n|^{p_i} \right) \]

by (14), \( \left( a_i(x, u_n, \nabla u_n) \right) \) and \( \left( H_i(x, u_n, \nabla u_n) \right) \) are bounded in \( L^{\overline{p}}(\Omega) \) then \( a_i(x, u_n, \nabla u_n) \rightarrow a_i(x, u, \nabla u) \) weakly in \( L^{\overline{p}}(\Omega) \) and \( H_i(x, u_n, \nabla u_n) \rightarrow H_i(x, u, \nabla u) \) weakly in \( L^{\overline{p}}(\Omega) \). Now we prove that \( g_i^n(x, u_n, \nabla u_n) \) is uniformly equi-integrable for \( i = 1, \ldots, N \). For any measurable \( E \) of \( \Omega \) and for any \( k \) in \( \mathbb{R}^* \), we have

\[ \int_{E} g_i^n(x, u_n, \nabla u_n) dx \leq \int_{E \cap [u_n \leq k]} g_i^n(x, u_n, \nabla u_n) dx + \int_{E \cap [u_n > k]} g_i^n(x, u_n, \nabla u_n) dx \]

\[ \leq \frac{1}{k} \int_{E \cap [u_n > k]} T_k(u_n)g_i^n(x, u_n, \nabla u_n) dx \]

we have \( g_i^n \) is uniformly equi-integrable for any \( i \), since \( g_i^n(x, u_n, \nabla u_n) \rightarrow g_i(x, u, \nabla u) \) a. e. in \( \Omega \), we get \( g_i^n(x, u_n, \nabla u_n) \rightarrow g_i(x, u, \nabla u) \) in \( L^1(\Omega) \). That allow us to pass to the limit in the approximate problem.

**Remark 1** The condition (6) can be substituted by

\[ \left| a_j(x, s, \xi) \right| \leq \frac{1}{j} \int_{[\xi]} \left( j_1 + |\xi| \right)^{p_i - 1} |\xi|, \]

where \( j_1 \) is a positive function in \( L^p(\Omega) \) for \( i = 1, \ldots, N \), and the condition (12) can be substituted by

\[ \left| g_j(x, s, \xi) \right| \leq C_i (\xi|\xi|^{p_i - 1}), \]

where \( C_i \) is a positive function in \( L^1(\Omega) \) for \( i = 1, \ldots, N \).

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References


