Doubly Nonlinear Parabolic Systems In Inhomogeneous Musielak-Orlicz-Sobolev Spaces

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1 Introduction

Given a bounded-connected open set \( \Omega \) of \( \mathbb{R}^N \) (\( N = 2 \)), with Lipschitz boundary \( \partial \Omega \), \( Q_T = \Omega \times (0, T) \) is the generic cylinder of an arbitrary finite height, \( T < +\infty \).

We prove the existence of a renormalized solutions for the nonlinear parabolic systems:

\[
d_{\partial t}(\varphi_i(x,u_i)) - \text{div}(a(x,t,u_i,Vu_i)) - \text{div}(\Phi_i(x,t,u_i)) + f_i(x,u_i,u_2) = 0 \quad \text{in } Q_T,
\]

\[
u_i = 0 \quad \text{on } (0,T) \times \partial \Omega,
\]

\[
b_i(x,u_i)(t=0) = b_i(x,u_{i0}) \quad \text{in } Q \Omega,
\]

where \( i = 1,2 \). Here the vector field \( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a Carathéodory function such that \(-\text{div}(a(x,t,u_i,Vu_i))\) is a Leray-Lions operator defined from the Inhomogeneous Musielak-Orlicz-Sobolev Spaces \( W^{1,\varphi}(Q_T) \) into its dual \( W^{-1,\varphi}(Q_T) \). Let \( b_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) a Carathéodory function such that for every \( x \in \Omega \), \( b_i(x,.) \) is a strictly increasing \( C^1 \)-function, the divergence term \( \Phi_i(x,t,u_i) \) is a Carathéodory function satisfy only a polynomial growth with respect to the anisotropic N-function \( \varphi \) (see (4.6)), the data \( u_{0,i} \) is in \( L^1(\Omega) \) such that \( b_i(.,u_{0,i}) \) in \( L^1(\Omega) \) and the source \( f_i \) is a Carathéodory function satisfy the assumptions (4.7)-(4.10).

When problem (1.1) is investigated, there is a difficulty due to the fact that the data \( b_i(x,u_{0,i}) \) and \( b_i(x,u_{0,2}) \) only belong to \( L^1 \) and the functions \( a(x,t,u_i,Vu_i), \Phi_i(x,t,u_i) \) and \( f_i(x,u_i,u_2) \) do not belong to \( L_{\infty}^1(Q_T) \) in general, so that proving existence of weak solution seems to be an arduous task, and we cannot use the Stocks formula in the a priori estimates of the nonlinearity \( \Phi_i(x,t,u_i) \). In order to overcome this difficulty, we work with the framework of renormalized solutions (see Definition 3.1). One of the models of applications of these operators is the system of Boussinesq:

\[
\frac{\partial u}{\partial t} + (u,\nabla)u - 2\text{div}(\mu(\theta)\varepsilon(u)) + \nabla p = F(\theta) \quad \text{in } Q_T
\]

\[
\frac{\partial b(\theta)}{\partial t} + u.\nabla b(\theta) - \Delta \theta = 2\mu(\theta)\varepsilon(u) \quad \text{in } Q_T
\]

\[
u(t=0) = u_0, \quad b(\theta)(t=0) = b(\theta_0) \quad \text{on } \Omega
\]

\[
u = 0, \quad \theta = 0 \quad \text{on } \partial \Omega \times (0,T)
\]

Equation first equation is the motion conservation equation, the unknowns are the fields of displacement \( u : Q_T \rightarrow \mathbb{R}^N \) and temperature \( \theta : Q_T \rightarrow \mathbb{R} \). The field \( \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \) is the strain rate tensor.

It is our purpose, in this paper to generalize the result of (1.1), (2.1), (3.1) and we prove the existence of a renormalized solution of system (1.1).

The plan of the paper is as follows: In Section 2 we give the framework space, in Section 3 and 4 we
gives some useful Lemmas and basic assumptions. In Section 5 we give the definition of a renormalized solution of \([1,1]\), and we establish (Theorem 5.1) the existence of such a solution.

2 Preliminaries

2.1 Musielak-Orlicz function

Let \(\Omega\) be an open subset of \(\mathbb{R}^N\) \((N \geq 2)\), and let \(\phi\) be a real-valued function defined in \(\Omega \times \mathbb{R}\) and satisfying conditions:

\[ \Phi_1: \phi(x,s) \text{ is an N-function for all } x \in \Omega \text{ (i.e.)} \]

\[ \Phi_2: \phi(x,t) \text{ is a measurable function for all } t \geq 0. \]

A function \(\phi\) which satisfies the conditions \(\Phi_1\) and \(\Phi_2\) is called a Musielak-Orlicz function.

For a Musielak-Orlicz function \(\phi\) we put \(\phi_1(t) = \phi(x,t)\) and we associate its non-negative reciprocal function \(\phi_1^{-1}\), with respect to \(t\), that is:

\[ \phi_1^{-1}(\phi(x,t)) = \phi(x,\phi_1^{-1}(t)) = t \]

Let \(\phi\) and \(\gamma\) be two Musielak-Orlicz functions, we say that \(\phi\) dominate \(\gamma\), and we write \(\phi \lesssim \gamma\), near infinity (resp.globally) if there exist two positive constants \(c\) and \(0\) such that for a.e. \(x \in \Omega\)

\[ \gamma(x,t) \leq \phi(x,ct) \text{ for all } t \geq 0 \text{ (resp. for all } t \geq 0). \]

We say that \(\gamma\) grows essentially less rapidly than \(\phi\) at 0 resp. near infinity and we write \(\gamma \ll \phi\), for every positive constant \(c\), we have

\[ \lim_{t \to 0} \sup_{x \in \Omega} \phi(x,ct) = 0 \]

Remark 2.1 [4]. If \(\gamma \ll \phi\) near infinity, then \(\forall \epsilon > 0\) there exist \(k(\epsilon) > 0\) such that for almost all \(x \in \Omega\), we have

\[ \gamma(x,t) \leq k(\epsilon)\phi(x,\epsilon t) \quad \forall t \geq 0 \]

2.2 Musielak-Orlicz space

For a Musielak-Orlicz function \(\phi\) and a measurable function \(u: \Omega \to \mathbb{R}\), we define the functional

\[ \rho_{\phi,\Omega}(u) = \int_{\Omega} \phi(x,|u(x)|)dx. \]

The set \(K_{\phi,\Omega}(\mathbb{R}) = \{u: \Omega \to \mathbb{R} \text{ measurable : } \rho_{\phi,\Omega}(u) < \infty\}\) is called the Musielak-Orlicz class. The Musielak-Orlicz space \(L^\phi(\Omega)\) is the vector space generated by \(K_{\phi,\Omega}(\mathbb{R})\); that is, \(L^\phi(\Omega)\) is the smallest linear space containing the set \(K_{\phi,\Omega}(\mathbb{R})\). Equivalently

\[ L^\phi(\Omega) = \{u: \Omega \to \mathbb{R} \text{ measurable : } \rho_{\phi,\Omega}(u)^\lambda < \infty, \text{ for some } \lambda > 0\}. \]

For any Musielak-Orlicz function \(\phi\), we put \(\psi(x,s) = \sup_{u \in L^\phi(\Omega)}(st - \phi(x,s))\).

\(\psi\) is called the Musielak-Orlicz function complementary to \(\phi\) (or conjugate of \(\phi\)) in the sense of Young with respect to \(s\).

We say that a sequence of function \(u_n \in L^\phi(\Omega)\) is modular convergent to \(u \in L^\phi(\Omega)\) if there exists a constant \(\lambda > 0\) such that \(\lim_{n \to \infty} \rho_{\phi,\Omega}(\frac{u_n - u}{\lambda}) = 0\).

This implies convergence for \(\sigma(\Pi L^\phi, \Pi L^\phi)\) [see [5]].

In the space \(L^\phi(\Omega)\), we define the following two norms

\[ \|u\|_\phi = \inf \{\lambda > 0 : \int_{\Omega} \phi(x,\frac{|u(x)|}{\lambda})dx \leq 1\} \]

which is called the Luxemburg norm, and the so-called Orlicz norm by

\[ \|u\|_{\phi,\Omega} = \sup_{\|v\|_\phi \leq 1} \int_{\Omega} |u(x)v(x)|dx \]

where \(\psi\) is the Musielak-Orlicz-Young function complementary to \(\phi\). These two norms are equivalent [8].

\[ K_{\phi,\Omega}(\mathbb{R}) = \{u: \Omega \to \mathbb{R} \text{ measurable : } D^\alpha u \in E_g(\Omega), \forall \alpha \leq 1\} \]

where \(\alpha = (\alpha_1, ..., \alpha_N), |\alpha| = |\alpha_1| + ...(\text{ for all } t \geq 0). \)

\(\forall \alpha \leq 1\). Let \(\phi_{\phi,\Omega}(u) = \sum_{|\alpha| \leq 1} \rho_{\phi,\Omega}(D^\alpha u)\) and \(\|u\|_{\phi,\Omega} = \inf_{\lambda > 0} \phi_{\phi,\Omega}(\frac{u}{\lambda}) \leq 1\) for \(u \in W^1L^\phi(\Omega)\).

These functionals are convex modular and a norm on \(W^1L^\phi(\Omega)\), respectively. Then pair \((W^1L^\phi(\Omega), \|u\|_{\phi,\Omega})\) is a Banach space if \(\phi\) satisfies the following condition [6].

There exists a constant \(c > 0\) such that \(\inf_{x \in \mathbb{R}} \phi(x,1) > c\)

The space \(W^1L^\phi(\Omega)\) is identified to a subspace of the product \(\prod_{\alpha \leq 1} L^\phi(\Omega) = \prod L^\phi\). We denote by \(D(\Omega)\) the Schwartz space of infinitely smooth functions with compact support in \(\Omega\) and by \(\sigma(\Omega)\) the restriction of \(D(\mathbb{R})\) on \(\Omega\). The space \(W^1_0L^\phi(\Omega)\) is defined as the \(\sigma(\Pi L^\phi, \Pi L^\phi)\) closure of \(D(\Omega)\) in \(W^1L^\phi(\Omega)\) and the space \(W^1_0E_g(\Omega)\) as the norm closure of the Schwartz space \(D(\Omega)\) in \(W^1L^\phi(\Omega)\). For two complementary Musielak-Orlicz functions \(\phi\) and \(\psi\), we have (see [7]).

- The Young inequality:

\[ \int_{\Omega} \phi(x,t)dx \leq \int_{\Omega} \phi(x,s)dx \]

for all \(s,t \geq 0, x \in \Omega\).

- The H"{o}lder inequality

\[ \int_{\Omega} \phi(x,t)dx \leq \|u\|_{\phi,\Omega} \|v\|_{\psi,\Omega} \]

for all \(u \in L^\phi(\Omega), v \in L^\psi(\Omega)\).
We say that a sequence of functions \( u_n \) converges to \( u \) for the modular convergence in \( W^{1,\infty}L^p(\Omega) \) (respectively in \( W^{1,\infty}L^p(\Omega) \)) if, for some \( \lambda > 0 \),

\[
\lim_{n \to \infty} \| u_n - u \|_{\Omega, \lambda} = 0
\]

The following spaces of distributions will also be used

\[
W^{-1}L^p(\Omega) = \{ f \in D'(\Omega) : f = \sum_{\alpha \leq 1} (-1)^\alpha D^\alpha f_{\alpha} \quad \text{where } f_{\alpha} \in L^p(\Omega) \}
\]

and

\[
W^{-1}E^\alpha(\Omega) = \{ f \in D'(\Omega) : f = \sum_{\alpha \leq 1} (-1)^\alpha D^\alpha f_{\alpha} \quad \text{where } f_{\alpha} \in E^\alpha(\Omega) \}
\]

2.3 Inhomogeneous Musielak-Orlicz-Sobolev spaces:

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \) and let \( Q = \Omega \times [0, T] \) with some given \( T > 0 \), let \( \varphi \) be a Musielak-Orlicz function. For each \( \alpha \in \mathbb{N}^N \), denote by \( D^\alpha \) the distributional derivative on \( QT \) of order \( \alpha \) with respect to the variable \( x \in \mathbb{R}^N \). The inhomogeneous Musielak-Orlicz-Sobolev spaces of order \( 1 \) are defined as follows

\[
W^{1,\infty}L^p(\Omega) = \{ u \in L^p(\Omega) : \forall \alpha \leq 1, \quad D^\alpha u \in L^p(\Omega) \}
\]

\[
W^{1,x}E^\alpha(\Omega) = \{ u \in E^\alpha(\Omega) : \forall \alpha \leq 1, \quad D^\alpha u \in E^\alpha(\Omega) \}
\]

The last is a subspace of the first one, and both are Banach spaces under the norm

\[
\| u \| = \sum_{|\alpha| \leq m} \| D^\alpha u \|_{p, \infty}
\]

We can easily show that they form a complementary system when \( \Omega \) is a Lipschitz domain. These spaces are considered as subspaces of the product space \( \Pi L^p(\Omega) \) which has \((N + 1)\) copies. We shall also consider the weak topologies \( \sigma(\Pi L^p, \Pi E^\alpha) \) and \( \sigma(\Pi L^p, \Pi E^\alpha) \). If \( u \in W^{1,\infty}L^p(\Omega) \) then the function \( t \mapsto u(t) = u(t, \cdot) \) is defined on \((0, T)\) with values in \( W^{1,\infty}L^p(\Omega) \). If, further, \( u \in W^{1,x}E^\alpha(\Omega) \) then this function is a \( W^{1,\infty}L^p(\Omega) \)-valued and is strongly measurable. Furthermore the following imbedding holds: \( W^{1,\infty}E^\alpha(\Omega) \subset L^1(0, T; W^{1,\infty}E^\alpha(\Omega)) \). The space \( W^{1,\infty}L^p(\Omega) \) is not in general separable, if \( u \in W^{1,\infty}L^p(\Omega) \), we cannot conclude that the function \( u(t) \) is measurable on \((0, T)\). However, the scalar function \( t \mapsto u(t) = \| u(t) \|_{p, \Omega} \) is in \( L^1(0, T) \). The space \( W^{1,\infty}E^\alpha(\Omega) \) is defined as the (norm) closure in \( W^{1,\infty}E^\alpha(\Omega) \) of \( D(\Omega) \). We can easily show that when \( \Omega \) is a Lipschitz domain then each element \( u \) of the closure of \( D(\Omega) \) with respect of the weak* topology \( \sigma(\Pi L^p, \Pi E^\alpha) \) is limit, in \( W^{1,\infty}L^p(\Omega) \), of some subsequence \( (u_i) \in D(\Omega) \) for the modular convergence, i.e. there exists \( \lambda > 0 \) such that for all \( |\alpha| \leq 1, \)

\[
\int_{\Omega} \varphi(x, t) \left( \frac{\| u_{\alpha} - u_{\alpha} \|_p}{\lambda} \right) dx dt \to 0 \quad \text{as } i \to \infty
\]

this implies that \( (u_i) \) converge to \( u \) in \( W^{1,\infty}L^p(\Omega) \) for the weak topology \( \sigma(\Pi L^p, \Pi E^\alpha) \). Consequently

\[
D(\Omega)_{\sigma(\Pi L^p, \Pi E^\alpha)} = D(\Omega)_{\sigma(\Pi L^p, \Pi E^\alpha)}
\]

this space will be denoted by \( W^{1,\infty}L^p(\Omega) \). Furthermore \( W^{1,\infty}E^\alpha(\Omega) = W^{1,\infty}L^p(\Omega) \cap \Pi E^\alpha \). We have the following complementary system \( F \) being the dual space of \( W^{1,\infty}L^p(\Omega) \). It is also, except for an isomorphism, the quotient of \( \Pi L^p \) by the polar set \( W^{1,\infty}L^p(\Omega) \), and will be denoted by \( F = W^{1,\infty}L^p(\Omega) \) and is shown that this space will be equipped with the usual quotient norm

where the inf is taken on all possible decompositions

The space \( F_0 \) is then given by \( F_0 = W^{-1,\infty}E^\alpha(\Omega) \).

Lemma 2.1 [4]. Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \) and let \( \varphi \) and \( \psi \) be two complementary Musielak-Orlicz functions which satisfy the following conditions:

- There exists a constant \( c > 0 \) such that
\[
\inf_{x \in \Omega} \varphi(x, 1) > c \tag{2.1}
\]

- \( \exists A > 0 \) such that for all \( x, y \in \Omega \) with \( |x - y| \leq \frac{1}{2} \), we have
\[
\frac{\varphi(x, t)}{\varphi(y, t)} \leq \left( \frac{A}{|x - y|^\alpha} \right) \quad \text{for all } t \geq 1 \tag{2.2}
\]

- \( \exists C > 0 \) such that \( \psi(y, t) \leq C \quad \text{a.e. in } \Omega \tag{2.4} \)

Under this assumptions \( D(\Omega) \) is dense in \( L^p(\Omega) \) with respect to the modular topology, \( D(\Omega) \) is dense in \( W^{1,\infty}L^p(\Omega) \) for the modular convergence and \( D(\Omega) \) is dense in \( W^{1,\infty}L^p(\Omega) \) for the modular convergence, consequently, the action of a distribution \( S \) in \( W^{-1,\infty}L^p(\Omega) \) on an element \( u \) of \( W^{1,\infty}L^p(\Omega) \) is well defined. It will be denoted by \( \langle S, u \rangle \).
2.4 Truncation Operator

$T_k, k > 0$, denotes the truncation function at level $k$ defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 2.2 [4]. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $\phi$ be an Musielak-Orlicz function and let $u \in W_0^1 L_\phi(\Omega)$ (resp. $u \in W^1 L_\phi(\Omega)$). Then $F(u) \in W^1 L_\phi(\Omega)$ (resp. $u \in W^1 L_\phi(\Omega)$). Moreover, if the set of discontinuity points $D$ of $F$ is finite, then

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(x) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega; \ u(x) \notin D\} \\ 0 & \text{a.e. in } \{x \in \Omega; \ u(x) \in D\} \end{cases}$$

Lemma 2.3 Suppose that $\Omega$ satisfies the segment property and let $u \in W_0^1 L_\phi(\Omega)$. Then, there exists a sequence $u_n \in D(\Omega)$ such that

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\phi(\Omega).$$

Furthermore, if $u \in W_1^1 L_\phi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N + 1)\|u\|_\infty$.

Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $\phi$ be a Musielak-Orlicz function satisfying:

$$\int_0^1 \phi_{-1}^{-1}(t) \frac{dt}{t} = \infty \text{ a.e. } x \in \Omega \tag{2.5}$$

and the conditions of Lemma 2.1. We may assume without loss of generality that

$$\int_0^1 \phi_{-1}^{-1}(t) \frac{dt}{t} < \infty \text{ a.e. } x \in \Omega \tag{2.6}$$

Define a function $\phi^* : \Omega \times [0, \infty) \to \mathbb{R}$ by $\phi^*(x, s) = \int_0^s \phi_{-1}^{-1}(t) dt \in \Omega$ and $s \in [0, \infty)$.

$\phi^*$ is called the Sobolev conjugate function of $\phi$ (see [1] for the case of Orlicz function).

Theorem 2.1 Let $\Omega$ be a bounded Lipschitz domain and let $\phi$ be a Musielak-Orlicz function satisfying 2.5 and the conditions of Lemma 2.1. Then

$$W_0^1 L_\phi(\Omega) \hookrightarrow L_{\phi^*}(\Omega)$$

where $\phi^*$ is the Sobolev conjugate function of $\phi$. Moreover, if $\phi$ is any Musielak-Orlicz function increasing essentially more slowly than $\phi^*$ near infinity, then the imbedding

$$W_0^1 L_\phi(\Omega) \hookrightarrow L_\phi(\Omega)$$

is compact.

Corollary 2.1 Under the same assumptions of theorem 5.1, we have

$$W_0^1 L_\phi(\Omega) \hookrightarrow L_{\phi^*}(\Omega)$$

Lemma 2.4 If a sequence $u_n \in L_\phi(\Omega)$ converges a.e. to $u$ and if $u_n$ remains bounded in $L_\phi(\Omega)$, then $u \in L_\phi(\Omega)$ and $u_n \to u$ for $a(L_\phi(\Omega), E_\phi(\Omega)).$

Lemma 2.5 Let $u_n, u \in L_\phi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $a(L_\phi(\Omega), L_\phi(\Omega))$.

See ([8]).

3 Technical lemma

Lemma 3.1 Under the assumptions of lemma 2.1 and by assuming that $\psi(x,t)$ decreases with respect to one of coordinate of $x$, there exists a constant $c_1 > 0$ which depends only on $\Omega$ such that

$$\int_\Omega \phi(x,|u|) dx \leq \int_\Omega \phi(x,c_1|\nabla u|) dx \tag{3.1}$$

Theorem 3.1 Let $\Omega$ be a bounded Lipschitz domain and let $\phi$ be a Musielak-Orlicz function satisfying the same conditions of Theorem 2.1. Then there exists a constant $\lambda > 0$ such that

$$\|u\|_\phi \leq \lambda \|\nabla u\|_\phi, \quad \forall \in W_0^1 L_\phi(\Omega)$$

4 Essential assumptions

Let $\Omega$ be an open subset of $\mathbb{R}^N (N \geq 2)$ satisfying the segment property, and let $\phi$ and $\gamma$ be two Musielak-Orlicz functions such that $\phi$ and its complementary $\psi$ satisfies conditions of Lemma 2.1 and $\gamma << \psi$.

$$b_i : \Omega \times \mathbb{R} \to \mathbb{R}$$

is a Carathéodory function such that for every $x \in \Omega$,

$$b_i(x,.)$$

is a strictly increasing $C^1(\mathbb{R})$-function and $b_i \in L^\infty(\Omega \times \mathbb{R})$ with $b_i(x,0) = 0$. Next for any $k > 0$, there exists a constant $A_k^i > 0$ and functions $A_k^i \in L^\infty(\Omega)$ and $B_k^i \in L_{\phi^*}(\Omega)$ such that:

$$A_k^i \leq \phi_{-1}^{-1} (\psi_{-1}^{-1}(\phi(x,v_2)(\xi)))$$

$$\beta > 0,$$

$$(\beta \circ x) \in E_\phi(\Omega),$$

$$(\beta) : (a(x,t,s,\xi) - a(x,s,\xi')(\xi - \xi')) > 0,$$

$$\Phi(x,\xi) \in \Omega \times \mathbb{R} \to L_\phi(\Omega)$$

is a Carathéodory function such that

$$\Phi(x,\xi) \in \Omega \times \mathbb{R} \to L_\phi(\Omega)$$

is a Carathéodory function with

$$f_1(x,0,s) = f_2(x,s,0) = 0 \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}.$$
and for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$,
\[
\text{sign}(s_1) f_i(x, s_1, s_2) \geq 0. \tag{4.8}
\]
The growth assumptions on $f_i$ are as follows: For each $K > 0$, there exists $\sigma_K > 0$ and a function $G_K$ in $L^1(\Omega)$ such that
\[
|f_i(x, s_1, s_2)| \leq F_i(x) + \sigma_K |b_i(x, s_1)| \tag{4.9}
\]
a.e. in $\Omega$, for all $s_1$ such that $|s_1| \leq K$, for all $s_2 \in \mathbb{R}$. For each $K > 0$, there exists $\lambda_K > 0$ and a function $G_K$ in $L^1(\Omega)$ such that
\[
|f_2(x, s_1, s_2)| \leq G_K(x) + \lambda_K |b_1(x, s_1)|, \tag{4.10}
\]
for almost every $x \in \Omega$, for every $s_2$ such that $|s_2| \leq K$, and for every $s_1 \in \mathbb{R}$. Finally, we assume the following condition on the initial data $u_{i,0}$: for $i=1,2$.

$u_{i,0}$ is a measurable function such that $b_i(\cdot, u_{i,0}) \in L^1(\Omega)$, \( \tag{4.11} \)

In this paper, for $K > 0$, we denote by $T_K : r \mapsto \min(K, \max(r, -K))$ the truncation function at height $K$. For any measurable subset $E$ of $\mathbb{Q}_T$, we denote by $\text{meas}(E)$ the Lebesgue measure of $E$. For any measurable function $v$ defined on $\mathbb{Q}$ and for any real numbers $s, x_{\alpha \in [0]})$ (respectively, $x_{\alpha \in [-1]}$, $x_{\alpha \in [1]}$) denote the characteristic function of the set $\{(x, t) \in \mathbb{Q}_T : v(t, x, t) < s\}$ (respectively, $\{(x, t) \in \mathbb{Q}_T : v(t, x, t) = s\}$, $\{(x, t) \in \mathbb{Q}_T : v(t, x, t) > s\}$).

**Definition 4.1** A couple of functions $(u_1, u_2)$ defined on $\mathbb{Q}$ is called a renormalized solution of $(4.1)-(4.11)$ if for $i=1,2$ the function $u_i$ satisfies

\[
T_K(u_i) \in W^{1, \infty}_{\text{loc}}(\mathbb{Q}_T) \quad \text{and} \quad b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega)), \tag{4.12}
\]

\[
\int_{m \leq |u_i| \leq m+1} a(x, t, u_i, \nabla u_i) \nabla u_i \, dx \, dt \to 0 \quad \text{as} \ m \to +\infty, \tag{4.13}
\]

For every function $S$ in $W^{2, \infty}(\mathbb{R})$ which is piecewise $C^1$ and such that $S'$ has a compact support, we have

\[
\frac{\partial B_{i,s}(x, u_i)}{\partial t} - \frac{\partial (S'(u_i)a(x, t, u_i, \nabla u_i))}{\partial t} + S''(u_i) a(x, t, u_i, \nabla u_i) \nabla u_i \nabla u_i
\]

\[+ \text{div}(S'(u_i) \phi_i(x, t, u_i)) - S''(u_i) \phi_i(x, t, u_i) \nabla u_i \nabla u_i
\]

\[+ f_i(x, u_i, u_2) S'(u_i) = 0, \tag{4.14}
\]

\[B_{i,s}(x, u_i)(t=0) = B_{i,s}(x, u_{i0}) \in \Omega, \tag{4.15}
\]

where $B_{i,s}(r) = \int_0^r b_i(x, s) S'(s) \, ds$.

Due to (4.12), each term in (4.14) has a meaning in $W^{-1, \infty}_{\text{loc}}(\mathbb{Q}_T) + L^1(\mathbb{Q}_T)$. Indeed, if $K$ such that $\sup \, S \subset [-K, K]$, the following identifications are made in (4.14).

- $B_{i,s}(x, u_i) \in L^\infty(\mathbb{Q}_T)$, since $|B_{i,s}(x, u_i)| \leq K \|a_i^0 \|_{L^\infty(\Omega)} \|S\|_{L^\infty(\mathbb{R})}.$
- $S(u_i)a(x, t, u_i, \nabla u_i)$ can be identified with $S'(u_i)a(x, t, T_K(u_i), \nabla T_K(u_i))$ a.e. in $\mathbb{Q}_T$. Since indeed $|T_K(u_i)| \leq K$ a.e. in $\mathbb{Q}_T$. As a consequence of (4.3), (4.12) and $S(u_i) \in L^\infty(\mathbb{Q}_T)$, it follows that

\[
S(u_i)a(x, T_K(u_i), \nabla T_K(u_i)) \in (L^\infty(\mathbb{Q}_T))^N.
\]
- $S(u_i)a(x, t, u_i, \nabla u_i)$ can be identified with $S'(u_i)a(x, t, T_K(u_i), \nabla T_K(u_i))$ a.e. in $\mathbb{Q}_T$ with (4.2) and (4.12) it has

\[
S(u_i)a(x, t, T_K(u_i), \nabla T_K(u_i)) \in L^1(\mathbb{Q}_T)
\]
- $S'(u_i)\Phi_i(u_i)$ and $S''(u_i)\Phi_i(u_i)\nabla u_i$ respectively identify with $S'(u_i)\Phi_i(T_K(u_i))$ and $S''(u_i)\Phi_i(T_K(u_i))\nabla T_K(u_i)$. In view of the properties of $S$ and (4.6), the functions $S', S''$ and $\Phi$ are bounded on $\mathbb{Q}$ so that (4.12) implies that $S'(u_i)\Phi_i(T_K(u_i)) \in (L^\infty(\mathbb{Q}_T))^N$ and $S''(u_i)\Phi_i(T_K(u_i))\nabla T_K(u_i) \in (L^\infty(\mathbb{Q}_T))^N$.
- $S'(u_i)f_i(x, u_1, u_2)$ identifies with $S'(u_i)f_i(x, T_K(u_1), u_2)$ a.e. in $\mathbb{Q}_T$ (or $S'(u_i)f_i(x, u_1, T_K(u_2))$ a.e. in $\mathbb{Q}_T$). Indeed, since $|T_K(u_i)| \leq K$ a.e. in $\mathbb{Q}_T$, assumptions (4.9) and (4.10) and using (4.12) and of $S(u_i) \in L^\infty(\mathbb{Q}_T)$, one has

\[
S'(u_i)f_i(x, T_K(u_i), u_2) \in L^1(\mathbb{Q}_T)
\]
and

\[
S''(u_i)f_i(x, T_K(u_1), T_K(u_2)) \in L^1(\mathbb{Q}_T).
\]

As consequence, (4.14) takes place in $D'(\mathbb{Q}_T)$ and that

\[
\frac{\partial B_{i,s}(x, u_i)}{\partial t} \in W^{-1, \infty}_{\text{loc}}(\mathbb{Q}_T) + L^1(\mathbb{Q}_T). \tag{4.16}
\]

Due to the properties of $S$ and (4.2),

\[
B_{i,s}(x, u_i) \in W^{1, \infty}_{\text{loc}}(\mathbb{Q}_T). \tag{4.17}
\]

Moreover (4.16) and (4.17) implies that $B_{i,s}(x, u_i) \in C^\infty([0, T]; L^1(\Omega))$ so that the initial condition (4.15) makes sense.

## 5 Existence result

We shall prove the following existence theorem

**Theorem 5.1** Assume that (4.1)-(4.11) hold true. There at least a renormalized solution $(u_1, u_2)$ of Problem (1.1).

We divide the proof in 5 steps.

**Step 1: Approximate problem.**

Let us introduce the following regularization of the data: for $n > 0$ and $i=1,2$

\[
b_{i,n}(x, s) = b_i(x, T_n(s)) + \frac{1}{n} \quad \forall s \in \mathbb{R}, \tag{5.1}
\]

\[
a_n(x, t, s, \xi) = a_n(x, T_n(s), \xi) a.e. \text{ in } \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N. \tag{5.2}
\]
\[\Phi_{i,n}(x,t,s) = \Phi_{i,n}(x,t,T_n(s)) \quad \text{a.e.} \quad (x,t) \in Q_T, \quad \forall s \in IR. \]

\[f_{1,n}(x,s_1,s_2) = f_1(x,T_n(s_1),s_2) \quad \text{a.e. in} \quad \Omega \forall s_1,s_2 \in IR, \quad \text{(5.3)}\]

\[f_{2,n}(x,s_1,s_2) = f_2(x,s_1,T_n(s_2)) \quad \text{a.e. in} \quad \Omega \forall s_1,s_2 \in IR, \quad \text{(5.4)}\]

\[u_{i,0n} \in C^0_0(\Omega), b_{i,n}(x,u_{i,0n}) \rightarrow b_i(x,u_{i,0}) \quad \text{in} \quad L^1(\Omega) \]

\[\text{as} \quad n \rightarrow +\infty \quad \text{(5.5)}\]

Let us now consider the regularized problem

\[\frac{\partial b_{i,n}(x,s)}{\partial s} \geq \frac{1}{n!} |b_{i,n}(x,s)| \leq \max |b_i(x,s)| + 1 \quad \forall s \in IR, \]

In view of \[(4.9), \quad \text{(4.10)}\]

\[f_{1,n} \quad \text{and} \quad f_{2,n} \quad \text{satisfy:} \quad \exists \ F_n \in L^1(\Omega), G_n \in L^1(\Omega) \quad \text{and} \quad a_n > 0, \lambda_n > 0, \quad \text{such that} \]

\[|f_{1,n}(x,s_1,s_2)| \leq F_n(x) + \sigma_n \max_{|s| \leq n} |b_i(x,s)| \quad \text{a.e. in} \quad x \in \Omega \forall s_1,s_2 \in IR, \]

\[|f_{2,n}(x,s_1,s_2)| \leq G_n(x) + \lambda_n \max_{|s| \leq n} |b_i(x,s)| \quad \text{a.e. in} \quad x \in \Omega \forall s_1,s_2 \in IR. \]

As a consequence, proving the existence of a weak solution \[u_{i,n} \in W^{1,\infty}_{0} L_{q}(Q_T) \]

is an easy task (see e.g. \[9\]).

\[\text{Step 2: A priori estimates.} \]

Let \(t \in (0, T)\) and using \(T_k(u_{i,n})\) as a test function in problem \[(5.7), \quad \text{(5.9)}\] we get:

\[\int_{Q_i} B^n_{i,k}(x,u_{i,0n}(t))dxdt + \int_{Q_i} f_{i,n}T_k(u_{i,n})dxdt \leq \int_{\Omega} B^n_{i,k}(x,u_{i,0n})dxdx, \]

where \(B^n_{i,k}(x,r) = \int_{0}^{r} \frac{\partial b_{i,n}(x,s)}{\partial s} T_k(s)ds\).

Due to definition of \(B^n_{i,k}\) we have:

\[\int_{\Omega} B^n_{i,k}(x,u_{i,0n}(t))dxdx \geq \frac{\lambda_n}{2} \int_{\Omega} |T_k(u_{i,n})|^2 dxdx, \quad \forall k > 0, \quad \text{(5.11)}\]

and

\[0 \leq \int_{\Omega} B^n_{i,k}(x,u_{i,0n})dxdx \leq k \int_{\Omega} |b_{i,n}(x,u_{i,0n})|dxdx \quad \text{(5.12)}\]

In view of \[(4.8), \quad \text{(4.9)}\] we have \[\int_{Q_i} f_{i,n}T_k(u_{i,n})dxdt \geq 0\]

Also, we obtain with Young inequality:

\[\int_{Q_t} \phi_{i,n}(x,t,u_{i,n})V T_k(u_{i,n})dxdtdt \quad \text{and} \quad \int_{[|u_{i,n}|<k]} \phi(x,T_k(u_{i,n}))dxdtdt \]

\[\leq \int_{[|u_{i,n}|<k]} \psi(x,\frac{1}{\sigma_0}) \phi_{i,n}(x,t,u_{i,n}))dxdtdt \]

\[+ \int_{[|u_{i,n}|<k]} \phi(x,\alpha_0^1 V T_k(u_{i,n}))dxdtdt \]

\[\leq \int_{[|u_{i,n}|<k]} \psi(x,\frac{1}{\alpha_0} \psi_{\max}^1 \phi(x,|k|))dxdtdt \]

\[+ \int_{[|u_{i,n}|<k]} \phi(x,\alpha_0^1 V T_k(u_{i,n}))dxdtdt \]

\[\leq \int_{Q_t} \phi_{i,n}(x,t,T_k(u_{i,n}))dxdtdt \quad \text{and} \quad \int_{Q_t} \phi(x,V T_k(u_{i,n}))dxdtdt \]

\[\leq C_{i,k} + \alpha_0^1 \int_{Q_t} \phi(x,V T_k(u_{i,n}))dxdtdt \quad \text{(5.13)}\]

We conclude that

\[\int_{\Omega} \frac{\lambda}{2} \int_{Q_t} |T_k(u_{i,n})|^2 dxdx + \alpha^1 \int_{Q_t} \phi(x,VT_k(u_{i,n}))dxdtdt \]

\[\leq \alpha_0^1 \int_{Q_t} \phi(x,VT_k(u_{i,n}))dxdx + C_{i,k} + k |b_i(x,u_{i,0n})| L^1(\Omega) \]

Then

\[\int_{\Omega} \frac{\lambda}{2} \int_{Q_t} |T_k(u_{i,n})|^2 dxdx \quad \text{(5.14)}\]

Choosing \(\alpha_0^1\) such that

\[0 < \alpha_0^1 < \min(1, \alpha^1)\]

we get

\[\int_{Q_t} \phi(x,VT_k(u_{i,n}))dxdtdt \leq C_{i,k} \quad \text{(5.15)}\]

Then, by \[(5.14), \quad \text{(5.15)}\] we conclude that \(T_k(u_{i,n})\) is bounded in \(W^{1,\infty}_{0} L_{q}(Q_T)\) independently of \(n\) and for any \(k \geq 0\), so there exists a subsequence still denoted by \(u_n\) such that

\[T_k(u_{i,n}) \rightarrow \psi_{t,k} \quad \text{(weakly in} \quad W^{1,\infty}_{0} L_{q}(Q_T) \quad \text{for} \quad \sigma(\Pi_{L_{q}}, \Pi_{E_{\phi}}) \quad \text{strongly in} \quad E_{\phi}(Q_T) \quad \text{and a.e. in} \quad Q_T). \]

Since Lemma \[3.3, \quad \text{and} \quad \text{(5.14)}\], we get also,

\[\phi(x,k) \leq \phi(|u_{i,n}| > k) \cap B_R \times [0,T] \quad \text{(min(1,} \quad \text{\alpha^1)}\]

\[\leq \int_{0}^{T} \int_{[|u_{i,n}|=k]} \phi(x,T_k(u_{i,n}))dxdtdt \]

\[\leq \int_{Q_T} \phi(x,T_k(u_{i,n}))dxdtdt \]
Consider now a function non decreasing $g_k \in C^2(\mathbb{R})$ such that $g_k(s) = s$ for $|s| \leq \frac{1}{2}$ and $g_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $g_k'(u_{i,n})$, we get

$$\frac{\partial g_k'(u_{i,n})}{\partial t} + a_n(x,t,u_{i,n})g_k'(u_{i,n}) \frac{\partial g_k'(u_{i,n})}{\partial x} + b_{i,n}(x,u_{i,n})g_k'(u_{i,n})V_T(u_{i,n})$$

$$+ \text{div} \left( \phi_{i,n}(x,t,u_{i,n}) g_k'(u_{i,n}) \psi_{i,n}(x,t,u_{i,n}) \right) + f_{i,n}(x,u_{i,n}) = 0 \quad \text{in} \quad D'(Q_T)$$

(5.20)

where $B_{i,k}(x,z) = \int_0^x \frac{\partial h_{i,k}(x,s)}{\partial s} g_k'(s)ds$.

Using (5.20), we can deduce that $g_k(u_{i,n})$ is bounded in $W^{1,\infty}(\Omega)$ and $\frac{\partial h_{i,k}(x,u_{i,n})}{\partial t}$ is bounded in $L^1(\Omega)$ independently of $n$. Thanks to (4.6) and properties of $g_k$, it follows that

$$\leq \|g_k'(\cdot)\|_{\infty} \|\psi^{-1}(\cdot)\|_{\infty} \int_{Q_T} \psi(x,T_k(u_{i,n}))dxdt$$

(5.21)

By (5.13), we get

$$\leq \|g_k'(\cdot)\|_{\infty} \|\psi^{-1}(\cdot)\|_{\infty} \int_{Q_T} \psi(x,T_k(u_{i,n}))dxdt$$

(5.22)

where $C_{i,k}$ and $C_{i,k}^2$ constants independently of $n$.

Proof of (5.16) and (5.17):
Consider now a function non decreasing $g_k \in C^2(\mathbb{R})$ such that $g_k(s) = s$ for $|s| \leq \frac{1}{2}$ and $g_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $g_k'(u_{i,n})$, we get

$$\frac{\partial g_k'(u_{i,n})}{\partial t} + a_n(x,t,u_{i,n})g_k'(u_{i,n}) \frac{\partial g_k'(u_{i,n})}{\partial x} + b_{i,n}(x,u_{i,n})g_k'(u_{i,n})V_T(u_{i,n})$$

$$+ \text{div} \left( \phi_{i,n}(x,t,u_{i,n}) g_k'(u_{i,n}) \psi_{i,n}(x,t,u_{i,n}) \right) + f_{i,n}(x,u_{i,n}) = 0 \quad \text{in} \quad D'(Q_T)$$

for almost any $t$ in $(0, T)$. Where, $b_{i,n}(x,u_{i,n}) \in L^\infty(0,T;L^1(\Omega))$. Indeed using $1 \leq T_k(u_{i,n})$ as a test function in (5.7),

$$\int_{Q_T} \phi_{i,n}(x,t,u_{i,n}) \psi_{i,n}(x,t,u_{i,n})dxdt = 0$$

(5.23)

By Young inequality and (4.6), we get

$$\leq \|g_k'(\cdot)\|_{\infty} \|\psi^{-1}(\cdot)\|_{\infty} \int_{Q_T} \psi(x,T_k(u_{i,n}))dxdt$$

(5.24)

where $C_{i,k}$ and $C_{i,k}^2$ constants independently of $n$. we conclude that $g_k(u_{i,n})$ is bounded in $L^1(\Omega) + W^{1,\infty}(\Omega)$ for $k \leq n$ which implies that $g_k(u_{i,n})$ is compact in $L^1(\Omega)$. Due to the choice of $g_k$, we conclude that for each $k$, the sequence $T_k(u_{i,n})$ converges almost everywhere in $Q_T$, which implies that the sequence $u_{i,n}$ converges almost everywhere to some measurable function $u_i$ in $Q_T$. Then by the same argument in [9], we have

$$u_{i,n} \rightarrow u_i \quad \text{a.e. in} \quad Q_T,$$  (5.21)

where $u_i$ is a measurable function defined in $Q_T$ and

$$b_{i,n}(x,u_{i,n}) \rightarrow b_i(x,u_{i}) \quad \text{a.e. in} \quad Q_T$$  (5.22)

weakly in $W^{1,\infty}(\Omega)$ for $|\phi_{i,n}|$ and $|\pi_{i,n}|$ strongly in $E_{\phi}(Q_T)$ and $a_i$ a.e in $Q_T$.

We now show that $b_i(x,u_{i}) \in L^\infty(0,T;L^1(\Omega))$. Indeed using $1 \leq T_k(u_{i,n})$ as a test function in (5.7),

$$\int_{Q_T} \phi_{i,n}(x,t,u_{i,n}) \psi_{i,n}(x,t,u_{i,n})dxdt = 0$$

(5.23)

By Young inequality and (4.6), we get

$$\leq \|g_k'(\cdot)\|_{\infty} \|\psi^{-1}(\cdot)\|_{\infty} \int_{Q_T} \psi(x,T_k(u_{i,n}))dxdt$$

(5.24)

where $C_{i,k}$ and $C_{i,k}^2$ constants independently of $n$. we conclude that $g_k(u_{i,n})$ is bounded in $L^1(\Omega) + W^{1,\infty}(\Omega)$ for $k \leq n$ which implies that $g_k(u_{i,n})$ is compact in $L^1(\Omega)$. Due to the choice of $g_k$, we conclude that for each $k$, the sequence $T_k(u_{i,n})$ converges almost everywhere in $Q_T$, which implies that the sequence $u_{i,n}$ converges almost everywhere to some measurable function $u_i$ in $Q_T$. Then by the same argument in [9], we have

$$u_{i,n} \rightarrow u_i \quad \text{a.e. in} \quad Q_T,$$  (5.21)

where $u_i$ is a measurable function defined in $Q_T$ and

$$b_{i,n}(x,u_{i,n}) \rightarrow b_i(x,u_{i}) \quad \text{a.e. in} \quad Q_T$$  (5.22)

weakly in $W^{1,\infty}(\Omega)$ for $|\phi_{i,n}|$ and $|\pi_{i,n}|$ strongly in $E_{\phi}(Q_T)$ and $a_i$ a.e in $Q_T$.
Using the Lebesgue’s Theorem and \( \varphi(x, VT_i(u_{i,n})) \in W^{1,1}_0(Q_T) \) in second term of the left hand side of the (5.25) and letting \( \varepsilon \to 0 \) in (5.24) we obtain
\[
\int_{\Omega} |b_{i,n}(x,u_{i,n})(t)| \, dx \leq \| b_{i,n}(x,u_{i,0,n}) \|_{L^1(\Omega)} \quad (5.26)
\]
for almost \( t \in (0,T) \), thanks to (5.6), (5.16), and passing to the limit-inf in (5.26), we obtain \( b_{i}(x,u_{i}) \in L^{\infty}(0,T;L^1(\Omega)) \). **Proof of 5.18:**
Following the same way in ([10]), we deduce that \( a_{n}(x,t,T_k(u_{i,n}),VT_k(u_{i,n})) \) is a bounded sequence in \( L^1_0((0,T)) \), and we obtain (5.18).

**Proof of 5.19:**
Multiplying the approximating equation (5.7) by the test function \( \theta_m(u_{i,n}) = T_{m+1}(u_{i,n}) - T_m(u_{i,n}) \)
\[
\int_{Q_T} B_{i,m}(x,u_{i,m}(T)) \, dx + \int_{Q_T} a_{n}(x,t,u_{i,n}) \nabla \theta_m(u_{i,n}) \, dx \, dt
\]
and strongly convergence in \( x \), we get
\[
\lim_{m \to +\infty} \int_{Q_T} \phi_{i,n}(x,t,u_{i,n}) \nabla \theta_m(u_{i,n}) \, dx \, dt = 0 \quad (5.28)
\]
and the other hand, we have
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \int_{Q_T} \phi_{i,n}(x,t,u_{i,n}) \nabla \theta_m(u_{i,n}) \, dx \, dt
\]
\[
\leq \lim_{m \to +\infty} \lim_{n \to +\infty} \int_{m \leq |u_{i,n}| \leq m+1} \phi_{i,n}(x,t,u_{i,n}) \, dx \, dt
\]
Using the pointwise convergence of \( u_{i,n} \) and by Lebesgue’s theorem, in the second term of the right side, we get
\[
\lim_{n \to +\infty} \int_{m \leq |u_{i,n}| \leq m+1} \phi_{i,n}(x,t,u_{i,n}) \, dx = 0 \quad (5.29)
\]
Then passing to the limit in (5.27), we get the (5.19).

**Step 3:** Let \( v_{i,j} \in D(Q_T) \) be a sequence such that \( v_{i,j} \to u_i \) in \( W^{1,1}_0(Q_T) \) for the modular convergence. This specific time regularization of \( T_k(v_{i,j}) \) (for fixed \( k \geq 0 \)) is defined as follows.

Let \( (\alpha_{i,0}^\mu) \) be a sequence of functions defined on \( \Omega \) such that
\[
\alpha_{i,0}^\mu \in L^\infty(\Omega) \cap W^{1,1}_0(\Omega) \quad \text{for all } \mu > 0
\]

and \( \alpha_{i,0}^\mu \) converges to \( T_k(u_{i,0}) \) a.e. in \( \Omega \) and \( \int_0^1 \| \alpha_{i,0}^\mu \|_{L^\infty(\Omega)} \, d\mu \) converges to \( 0 \) as \( \mu \to +\infty \). For \( k \geq 0 \) and \( \mu > 0 \), let us consider the unique solution \( (T_k(v_{i,j}))_{\mu} \in L^\infty(Q) \cap W^{1,1}_0(Q_T) \) of the monotone problem:
\[
\frac{\partial (T_k(v_{i,j}))_{\mu}}{\partial t} + \mu (T_k(v_{i,j}))_{\mu} - T_k(v_{i,j}) = 0 \quad \text{in } D'(\Omega),
\]
\[
(T_k(v_{i,j}))_{\mu}(t = 0) = \alpha_{i,0}^\mu \quad \text{in } \Omega.
\]
Remark that due to
\[
\frac{\partial (T_k(v_{i,j}))_{\mu}}{\partial t} \in W^{1,1}_0(Q_T)
\]
We just recall that, for fixed $K \geq 0$:

\[
\liminf_{\mu \to +\infty} \lim_{j \to +\infty} \limsup_{n \to +\infty} \int_{Q_T} \left( \frac{\partial b_i(x,u^n)}{\partial t} + S'_m(u^n,\nabla W_{i,j,p}^n) \right) dx \, dt = 0.
\]

(5.41)

Next we pass to the limit as $n \to +\infty$, $j \to +\infty$, $\mu \to +\infty$ and then $m \to +\infty$, the real number $K \geq 0$ being kept fixed. In order to perform this task we prove below the following results for the modular convergence as $j \to +\infty$.

\[
(T_k(u_j))_\mu \to T_k(u_m) \quad \text{in} \quad W_0^1L_\psi(Q_T)
\]

(5.35)

for the modular convergence as $\mu \to +\infty$.

\[
(T_k(u_i))_\mu \to T_k(u_i) \quad \text{in} \quad W_0^1L_\psi(Q_T)
\]

(5.36)

Through setting, for fixed $K \geq 0$,

\[
W_{i,j,p}^n = T_k(u_{i,n}) - T_k(u_j)_\mu \quad \text{and} \quad W_{i,m}^n = T_k(u_{i,m}) - T_k(u_i)_\mu
\]

(5.39)

we obtain upon integration,

\[
\begin{align*}
\int_{Q_T} & \left( \frac{\partial b_{i,n}(u_{i,m})}{\partial t} + S'_m(u_{i,m},\nabla W_{i,j,p}^n) \right) dx \, dt \\
& + \int_{Q_T} S'_m(u_{i,m})a_{i,n}(x,u_{i,m},\nabla u_{i,m}) \nabla W_{i,j,p}^n dx \, dt \\
& + \int_{Q_T} S''_{m}(u_{i,m})W_{i,j,p}^n a_{n}(x,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} dx \, dt \\
& + \int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})S'_m(u_{i,m},\nabla W_{i,j,p}^n) dx \, dt \\
& + \int_{Q_T} S''_{m}(u_{i,m})W_{i,j,p}^n \Phi_{i,n}(x,t,u_{i,n}) \nabla u_{i,n} dx \, dt \\
& + \int_{Q_T} f_{i,n}(x,u_{i,n},u_{2,n})S'_m(u_{i,m},\nabla W_{i,j,p}^n) dx \, dt = 0
\end{align*}
\]

(5.40)

Next we pass to the limit as $n \to +\infty$, $j \to +\infty$, $\mu \to +\infty$ and then $m \to +\infty$, the real number $K \geq 0$ being kept fixed. In order to perform this task we prove below the following results for the modular convergence as $j \to +\infty$.

\[
(T_k(u_i))_\mu \to T_k(u_j) \quad \text{in} \quad W_0^1L_\psi(Q_T)
\]

(5.37)

Now, we introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions $S_m$ such that, for any $m \geq 1$

\[
S_m(r) = r \text{ for } |r| \leq m, \quad \supp(S_m) \subset \left([-m+1),(m+1)\right],
\]

(5.38)

\[
\|S'_m\|_{L^\infty(\mathbb{R})} \leq 1.
\]

(5.39)

Proof of (5.41):

Lemma 5.1

\[
\int_{Q_T} \left( \frac{\partial b_{i,n}(x,u_{i,m})}{\partial t} + S'_m(u_{i,m},\nabla W_{i,j,p}^n) \right) dx \, dt \geq \epsilon(n,j,\mu,m).
\]

(5.46)

See [23]. Proof of (5.42):

If we take $n > m + 1$, we get

\[
\psi_i(x,t,u_{i,m})S'_m(u_{i,m},\nabla W_{i,j,p}^n) = \psi_i(x,T_{m+1}(u_{i,m}))S'_m(u_{i,m}).
\]

Using (4.6), we have:

\[
\psi_i(x,t,u_{i,m})S'_m(u_{i,m},\nabla W_{i,j,p}^n) \leq (m+1)\psi_i(x,T_{m+1}(u_{i,m}))
\leq (m+1)\psi_i(\|x(t)|\|_{L^\infty(Q_T)}) \leq M(m+1)
\]

Then $\psi_i(x,t,u_{i,m})S'_m(u_{i,m},\nabla W_{i,j,p}^n)$ is bounded in $L_\psi(Q_T)$, thus, by using the pointwise convergence of $u_{i,m}$ and Lebesgue’s theorem we obtain $\psi_i(x,t,u_{i,m})S'_m(u_{i,m},\nabla W_{i,j,p}^n)$ with the modular convergence as $n \to +\infty$, then $\lim_{n \to +\infty} \psi_i(x,t,u_{i,m})S'_m(u_{i,m},\nabla W_{i,j,p}^n)$ for $\|\psi_i\|_{L_\psi} \leq 1$, for convergence to $\nabla T_k(u_m) - \nabla T_k(u_j)$ weakly in $(L_\psi(Q_T))'$,

\[
\int_{Q_T} \psi_i(x,t,u_{i,m})S'_m(u_{i,m},\nabla W_{i,j,p}^n) dx \, dt \to 0
\]

(5.47)
as \( n \to +\infty \).

By using the modular convergence of \( W_{i,j,n} \) as \( j \to +\infty \) and letting \( \mu \) tends to infinity, we get (5.42).

**Proof of (5.43):**

For \( n > m = 1 > k \), we have \( \nabla u_{i,n} S_{m}^{
u}(u_{i,n}) = \nabla T_{m+1}(u_{n}) \) a.e. in \( Q_{T} \). By the almost everywhere convergence of \( u_{i,n} \) we have \( W_{i,j,n}^{ \mu } \to W_{i,j,\mu} \) in \( L^{\infty}(Q_{T}) \) weakly* and since the sequence \( \phi_{i,n}(x,t,T_{m+1}(u_{n})) \) converges strongly in \( E_{\phi}(Q_{T}) \) then

\[
\phi_{i,n}(x,t,T_{m+1}(u_{n})) W_{i,n}^{ \mu } \to \phi_{i,n}(x,t,T_{m+1}(u_{1})) W_{i,j,\mu}
\]

converge strongly in \( E_{\phi}(Q_{T}) \) as \( n \to +\infty \). By virtue of \( \nabla T_{m+1}(u_{n}) \to \nabla T_{m+1}(u_{1}) \) weakly in \( (L_{\phi}(Q_{T}))^{N} \) as \( n \to +\infty \) we have

\[
\int_{|u_{i}| \leq m}^{\mu} \phi(x,t,u_{1}) \nabla u_{i} W_{i,j,\mu}^{\nu} \, dx \, dt
\]

as \( n \to +\infty \) with the modular convergence of \( W_{i,j,\mu} \) as \( j \to +\infty \) and letting \( \mu \to +\infty \) we get (5.43).

**Proof of (5.44):**

For any \( m \geq 1 \) fixed, we have

\[
\int_{Q_{T}} S_{m}^{
u}(u_{i,n}) a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} W_{i,j,\mu}^{\nu} \, dx \, dt
\]

\[
\leq \|S_{m}^{
u}\|_{L^{\infty}(Q_{T})} \|\nabla u_{i,n}\|_{L^{\infty}(Q_{T})} \int_{|u_{i}| \leq m} a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} \, dx \, dt
\]

for any \( m \geq 1 \), and any \( \mu > 0 \). In view (5.37) and (5.38), we can obtain

\[
\limsup_{n \to +\infty} \int_{Q_{T}} S_{m}^{
u}(u_{i,n}) a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} W_{i,j,\mu}^{\nu} \, dx \, dt
\]

\[
\leq 2K \limsup_{n \to +\infty} \int_{|u_{i}| \leq m} a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} \, dx \, dt
\]

for any \( m \geq 1 \). Using (5.19) we pass to the limit as \( m \to +\infty \) in (5.50) and we obtain (5.44).

**Proof of (5.45):**

For fixed \( n \geq 1 \) and \( n > m + 1 \), we have

\[
f_{1,n}(x,u_{1,n},u_{2,n}) S_{m}^{
u}(u_{i,n})
\]

\[
= f_{1}(x,T_{m+1}(u_{1,n}),T_{m+1}(u_{2,n})) S_{m}^{
u}(u_{i,n})
\]

\[
f_{2,n}(x,u_{1,n},u_{2,n}) S_{m}^{
u}(u_{2,n})
\]

\[
= f_{2}(x,T_{m+1}(u_{1,n}),T_{m+1}(u_{2,n})) S_{m}^{
u}(u_{2,n})
\]

In view (4.9), (4.10), (5.22) and Lebesgue’s theorem allow us to get, for

\[
\lim_{n \to +\infty} \int_{Q_{T}} f_{1,n}(x,u_{1,n},u_{2,n}) S_{m}^{
u}(u_{i,n}) W_{i,j,\mu}^{\nu} \, dx \, dt
\]

\[
= \int_{Q_{T}} f_{1}(x,u_{1},u_{2}) S_{m}^{
u}(u_{1}) W_{i,j,\mu}^{\nu} \, dx \, dt
\]

Using (5.35), we follow a similar way we get as \( j \to +\infty \)

\[
\lim_{j \to +\infty} \int_{Q_{T}} f_{1}(x,u_{1},u_{2}) S_{m}^{
u}(u_{1}) W_{i,j,\mu}^{\nu} \, dx \, dt
\]

\[
= \int_{Q_{T}} f_{1}(x,u_{1},u_{2}) S_{m}^{
u}(u_{1}) (T_{K}(u_{1}) - T_{K}(u_{1})_{\mu}) \, dx \, dt
\]

we fixed \( m > 1 \), and using (5.36), we have

\[
\lim_{\mu \to +\infty} \int_{Q_{T}} f_{1}(x,u_{1},u_{2}) S_{m}^{
u}(u_{1}) (T_{K}(u_{1}) - T_{K}(u_{1})_{\mu}) \, dx \, dt = 0
\]

Then we conclude the proof of (5.45).

**Proof of (5.46):**

If we pass to the lim-sup when \( n,j \) and \( \mu \) tends to +\( \infty \) and then to the limit as \( m \) tends to +\( \infty \) in (5.40). We obtain using (5.41)–(5.45), for any \( K \geq 0 \),

\[
\lim_{m \to +\infty} \limsup_{\mu \to +\infty} \int_{Q_{T}} S_{m}^{
u}(u_{i,n}) a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_{K}(u_{i,n}) - \nabla T_{K}(u_{i,n})_{\mu} \, dx \, dt \leq 0.
\]

Since

\[
S_{m}^{
u}(u_{i,n}) a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_{K}(u_{i,n}) - a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_{K}(u_{i,n})_{\mu}
\]

for \( n > K \) and \( K \leq m \). Then, for \( K \leq m \),

\[
\limsup_{n \to +\infty} \int_{Q_{T}} a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_{K}(u_{i,n}) \, dx \, dt
\]

\[
\leq \limsup_{n \to +\infty} \int_{Q_{T}} a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_{K}(u_{i,n})_{\mu} \, dx \, dt
\]

Thanks to (5.38), we have in the right hand side of (5.51), for \( n > m + 1 \),

\[
S_{m}^{
u}(u_{i,n}) a_{n}(x,t,u_{i,n},\nabla u_{i,n}) - S_{m}^{
u}(u_{i,n}) a_{n}(x,t,u_{i,n},\nabla u_{i,n})
\]

\[
= S_{m}^{
u}(u_{i,n}) a_{n}(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})) \, dx \, dt
\]

Using (5.18), and fixing \( m \geq 1 \), we get

\[
S_{m}^{
u}(u_{i,n}) a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \to S_{m}^{
u}(u_{i,n}) X_{i,m+1} \text{ weakly in } (L_{\phi}(Q_{T}))^{N}
\]

when \( n \to +\infty \).

We pass to limit as \( j \to +\infty \) and \( \mu \to +\infty \), and using (5.35)–(5.36)

\[
\limsup_{m \to +\infty} \limsup_{j \to +\infty} \limsup_{\mu \to +\infty} \int_{Q_{T}} S_{m}^{
u}(u_{i,n}) a_{n}(x,t,u_{i,n},\nabla u_{i,n}) \nabla T_{K}(u_{i,n}) \, dx \, dt
\]

\[
= \int_{Q_{T}} S_{m}^{
u}(u_{i,n}) X_{i,m+1} \nabla T_{K}(u_{i,n}) \, dx \, dt
\]

\[
= \int_{Q_{T}} X_{i,m+1} \nabla T_{K}(u_{i,n}) \, dx \, dt
\]
where \( K \leq m \), since \( S'_n(r) = 1 \) for \( |r| \leq m \).

On the other hand, for \( K \leq m \), we have
\[
a(x, t, T_{m+1}(u_{i,m}), \nabla T_{m+1}(u_{i,m}))x_{|u_{i,n}| < K}
= a(x, t, T_K(u_{i,n}), \nabla T_K(u_{i,n}))x_{|u_{i,n}| < K},
\]
a.e. in \( Q_T \). Passing to the limit as \( n \to +\infty \), we obtain
\[
X_{i,m+1} \nabla T_K(u_i) = X_K \nabla T_K(u_i) \quad \text{a.e. in } Q_T. \tag{5.54}
\]

Then we obtain \( \text{(5.46)} \).

**Proof of (5.48):**

Let \( K \geq 0 \) be fixed. Using (4.5) we have
\[
a(x, t, T_K(u_{i,n}), \nabla T_K(u_{i,n})) - a(x, t, T_K(u_{i,n}), \nabla T_K(u_i))
\]
\[
\int_{Q_T} \left[ \nabla T_K(u_{i,n}) - \nabla T_K(u_i) \right] dx dt \geq 0, \tag{5.55}
\]
weakly in \( L^1(Q_T) \),

In view of (1.1) and (5.22), we get
\[
a(x, t, T_K(u_{i,n}), \nabla T_K(u_{i,n})) \to a(x, t, T_K(u_i), \nabla T_K(u_i)) \quad \text{a.e. in } Q_T,
\]
as \( n \to +\infty \), and by (4.2) and Lebesgue’s theorem, we obtain
\[
a(x, t, T_K(u_{i,n}), \nabla T_K(u_i)) \to a(x, t, T_K(u_i), \nabla T_K(u_i))
\]
strongly in \( (L_\infty(Q_T))^N \). Using (5.46), (5.22), (5.18) and (5.55), we can pass to the lim-sup as \( n \to +\infty \) in (5.55) to obtain (5.48).

To finish this step, we prove this Lemma:

**Lemma 5.2** For \( i = 1, 2 \) and fixed \( K \geq 0 \), we have
\[
X_{i,K} = a(x, t, T_K(u_i), \nabla T_K(u_i)) \quad \text{a.e. in } Q. \tag{5.57}
\]

Also, as \( n \to +\infty \),
\[
a(x, t, T_K(u_{i,n}), \nabla T_K(u_{i,n})) \nabla T_K(u_{i,n}) \to a(x, t, T_K(u_i), \nabla T_K(u_i)) \nabla T_K(u_i)
\]
weakly in \( L^1(Q_T) \).

**Proof of (5.57):**

It’s easy to see that
\[
a_n(x, t, T_K(u_{i,n}), \xi) = a(x, t, T_K(u_{i,n}), \xi) = a(x, t, T_K(u_{i,n}), \xi)
\]
a.e. in \( Q_T \) for any \( K \geq 0 \), any \( n > K \) and any \( \xi \in \mathbb{R}^N \).

In view of (5.18), (5.48) and (5.56), we obtain
\[
\lim_{n \to +\infty} \int_{Q_T} a_n(x, t, T_{m+1}(u_{i,m}), \nabla T_{m+1}(u_{i,m})) \nabla T_{m+1}(u_{i,m}) dx dt
= \int_{Q_T} X_{i,K} \nabla T_K(u_i) dx dt. \tag{5.59}
\]

Since (1.1), (4.4) and (5.22), imply that the function
\[
a_K(x, s, \xi) \quad \text{is continuous and bounded with respect to } s.
\]

Then we conclude that (5.57).

**Proof of (5.58):**

Using (4.5) and (5.48), for any \( K \geq 0 \) and any \( T' < T \), we have
\[
\left[ a(x, t, T_K(u_{i,n}), \nabla T_K(u_{i,n})) - a(x, t, T_K(u_i), \nabla T_K(u_i)) \right]
\times \nabla T_K(u_{i,n}) - \nabla T_K(u_i) \to 0 \tag{5.60}
\]
strongly in \( L^1(Q_T) \) as \( n \to +\infty \).

On the other hand with (5.22), (5.18), (5.56) and (5.57), we get
for \(n > m + 1\). According to (5.58), one can pass to the limit as \(n \to +\infty\); for fixed \(m \geq 0\) to obtain
\[
\lim_{n \to \infty} \int_{m \leq |u_n| \leq m+1} a_n(x,t,u_{i,n},\nabla u_{i,n}) \, dx \, dt = \int_{Q} \left( a(x,t,T_{m+1}^{-1}(u_i)) \nabla T_{m+1}^{-1}(u_i) \right) \, dx \, dt
\]
\[
- \int_{Q} a(x,t,T_{m+1}^{-1}(u_i)) \nabla T_{m+1}^{-1}(u_i) \, dx \, dt
\]
\[
= \int_{m \leq |u_i| \leq m+1} a(x,t,u_i,\nabla u_i) \, dx \, dt
\]
Pass to limit as \(m \to \infty\) and using (5.19) show that \(u_i\) satisfies (4.13).

Now we show that \(u_i\) to satisfy (4.14) and (4.15).

Let \(S\) be a function in \(W^{2,\infty}(\mathbb{R})\) such that \(S\) has a compact support. Let \(K\) be a positive real number such that supp \(S' \subset [-K,K]\). The Pointwise multiplication of the approximate equation (1.1) by \(S'(u_{i,n})\) leads to
\[
\frac{\partial B_{i,n}^u(S'(u_{i,n}))}{\partial t} - \text{div} \left( S'(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \right) + S''(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) - \text{div} \left( S'(u_{i,n}) B_{i,n}^u(x,t,u_{i,n}) \right) + \Phi_{i,n}^u(x,t,u_{i,n}) = f_{i,n}(x,t,u_{i,n},u_{i,n}^n) S'(u_{i,n})
\]
in \(D'(Q_T)\), for \(i = 1,2\).

Now we pass to the limit in each term of (5.63).

Limit of \(\frac{\partial B_{i,n}^u(S'(u_{i,n}))}{\partial t}\). Since \(B_{i,n}^u(S'(u_{i,n}))\) converges to \(B_{i,n}^u(u_i)\) a.e. in \(Q_T\) and in \(L^\infty(Q_T)\) weakly * and \(S\) is bounded and continuous. Then \(\frac{\partial B_{i,n}^u(S'(u_{i,n}))}{\partial t}\) converges to \(\frac{\partial B_{i,n}^u(S'(u_i))}{\partial t}\) in \(D'(Q_T)\) as \(n \to +\infty\).

Limit of \(\text{div} \left( S'(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \right)\): Since \(\text{supp} \, S' \subset [-K,K]\), for \(n > K\), we have
\[
S'(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) = S'(u_{i,n}) a_n(x,t,T_{K}^{-1}(u_i),\nabla T_{K}^{-1}(u_i))
\]
a.e. in \(Q_T\). Using the pointwise convergence of \(u_{i,n}\) in (5.38), (5.18) and (5.57), imply that
\[
S'(u_{i,n}) a_n(x,t,T_{K}^{-1}(u_i),\nabla T_{K}^{-1}(u_i)) \to S'(u_i) a_n(x,t,T_{K}^{-1}(u_i),\nabla T_{K}^{-1}(u_i))
\]
weakly in \((L^p(Q_T))^N\), for \(p(\Pi \wedge \Pi \wedge \Pi)\) as \(n \to +\infty\), since \(S'(u_i) = 0\) for \(|u_i| \geq K\) a.e. in \(Q_T\). And
\[
S'(u_i) a_n(x,t,T_{K}^{-1}(u_i),\nabla T_{K}^{-1}(u_i)) = S'(u_i) a_n(x,t,u_i,\nabla u_i)
\]
a.e. in \(Q_T\).

Limit of \(S''(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n}\). Since \(\text{supp} \, S'' \subset [-K,K]\), for \(n > K\), we have
\[
S''(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} \to S''(u_i) a_n(x,t,T_{K}^{-1}(u_i),\nabla T_{K}^{-1}(u_i)) \nabla T_{K}^{-1}(u_i)\quad \text{a.e. in } Q_T.
\]
The pointwise convergence of \(S''(u_{i,n})\) to \(S''(u_i)\) as \(n \to +\infty\), (5.38) and (5.58) we have
\[
S''(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} \to S''(u_i) a_n(x,t,T_{K}^{-1}(u_i),\nabla T_{K}^{-1}(u_i)) \nabla T_{K}^{-1}(u_i)
\]
weakly in \(L^1(Q_T)\), as \(n \to +\infty\), and
\[
S''(u_i) a_n(x,t,T_{K}^{-1}(u_i),\nabla T_{K}^{-1}(u_i)) \nabla T_{K}^{-1}(u_i) \to S''(u_i) a(x,t,u_i,\nabla u_i) \nabla u_i\quad \text{a.e. in } Q_T.
\]

Limit of \(S'(u_{i,n}) \Phi_{i,n}^u(x,t,u_{i,n})\): We have
\[
S'(u_{i,n}) \Phi_{i,n}^u(x,t,u_{i,n}) = S'(u_{i,n}) \Phi_{i,n}^u(x,t,T_{K}^{-1}(u_i)) \nabla T_{K}^{-1}(u_i)
\]
a.e. in \(Q_T\). Since \(\text{supp} \, S' \subset [-K,K]\) Using (4.5), (5.24) and (5.16), it’s easy to see that
\[
S'(u_{i,n}) \Phi_{i,n}^u(x,t,u_{i,n}) \to S'(u_i) \Phi(x,t,T_{K}^{-1}(u_i))
\]
weakly for \(\sigma(\Pi \wedge \Pi \wedge \Pi)\) as \(n \to +\infty\). And \(S'(u_i) \Phi(x,t,T_{K}^{-1}(u_i)) = S'(u_i) \Phi(x,t,u_i)\) a.e. in \(Q_T\).

Limit of \(f_{i,n}(x,u_{i,n},u_{i,n}^n) S'(u_{i,n})\): Since \(S' \in W^{1,\infty}(\mathbb{R})\) with \(\text{supp} \, S' \subset [-K,K]\), we have \(f_{i,n}(x,u_{i,n},u_{i,n}^n) \to f_i(x,t,u_i,\nabla S' (u_i))\) a.e. in \(Q_T\). The weakly convergence of truncation allows us to prove that
\[
S''(u_{i,n}) \Phi_{i,n}^u(x,t,u_{i,n}) \nabla u_{i,n} \to f_i(x,t,u_i,\nabla S' (u_i)),
\]
strongly in \(L^1(Q_T)\).

Limit of \(f_{i,n}(x,u_{i,n},u_{i,n}^n) S'(u_{i,n})\): Using (4.9), (4.10), (5.4) and (5.5), we have
\[
f_{i,n}(x,u_{i,n},u_{i,n}^n) S'(u_{i,n}) \to f_i(x,t,u_i,\nabla S' (u_i))
\]
strongly in \(L^1(Q_T) + W^{1,\infty}(\mathbb{R})\). As a consequence, an Aubin’s type Lemma (see e.g., [11], Corollary 4) implies that \(B_q(x,u_{i,n})\) lies in a compact set of \(C^0([0,T];L^1(\Omega))\).

It follows that, on one hand, \(B_q(x,u_{i,n}) (t = 0)\) converges to \(B_q(x,u_i) (t = 0)\) strongly in \(L^1(\Omega)\). On the other hand, the smoothness of \(B_q\) imply that \(B_q(x,u_{i,n}) (t = 0)\) converges to \(B_q(x,u_i) (t = 0)\) strongly in \(L^1(\Omega)\), we conclude that \(B_q(x,u_{i,n}) (t = 0) = B_q(x,u_{i,0})\) converges to \(B_q(x,u_i) (t = 0)\) strongly in \(L^1(\Omega)\), we obtain \(B_q(x,u_i) (t = 0) = B_q(x,u_{i,0})\) a.e. in \(\Omega\) and for all \(M > 0\), now letting \(M \to +\infty\), we conclude that \(b(x,u_i) (t = 0) = b(x,u_{i,0})\) a.e. in \(\Omega\).

As a conclusion, the proof of Theorem 5.1 is complete.
References