Multi-Step Iteration Algorithm of Total Asymptotically Quasi-Nonexpansive Maps
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ARTICLE INFO
Article history: Received: 14 February 2019 Accepted: 28 April, 2019 Online: 21 May, 2019

Keywords: Banach space, total asymptotically quasi-nonexpansive map, weak convergence, strong convergence, common fixed points

ABSTRACT
In Banach spaces an iteration algorithm for two finite families of total asymptotically quasi-nonexpansive maps is introduced. Weak and strong convergence theorems of this algorithm to approximation common fixed points are proved by using suitable conditions. As well as, numerical example by using Mat-lab is given.

1. Introduction and Preliminaries
This paper was originally published in the Conference: 2018 International Conference on Advanced Science and Engineering (ICOASE), Iraq [1]. It is well known that the concept of asymptotically nonexpansive introduced by Goebel and Kirk [2]. Additionally, every asymptotically nonexpansive map of a Banach space has a fixed point is proved. In [3], Petryshyn and Williamson proved the weak and strong convergence for quasi-nonexpansive map by using a sufficient and necessary condition. Alber [4], a new class of asymptotically nonexpansive is introduced. As well as, approximating methods for finding their fixed points are studied. In 2014, G. S. Saluja [5] established the strong and weak convergence for approximating common fixed point for generalized asymptotically quasi-nonexpansive maps in a Banach space.

Very recently, In [6], the authors proposed an implicit iteration for two finite families of generalized asymptotically quasi-nonexpansive maps. As well as, some strong convergence theorems are established. It is useful to point out our findings in this area which appeared in [7].

Let B be a non-empty closed convex subset of a real Banach space M and T be a self-map of B. The set of all fixed points denoted by F(T). A self-map T from B into M is called nonexpansive map [2] if

\[ \|Ta - Tb\| \leq \|a - b\| \text{ for all } a, b \in B \]

and is called quasi-nonexpansive map [6] if \( F(T) \neq \emptyset \) and

\[ \|Ta - a'\| \leq \|a - a'\| \]

for all \( a \in B \) and for all \( a' \in F(T) \).

A Banach space M is satisfying:

- "Opial’s condition if for each sequence \( (a_n) \) in M, it is weak convergence to a implies that \( \lim_{n \to \infty} \inf \|a_n - a\| < \lim_{n \to \infty} \inf \|a_n - b\| \) for all \( b \in M \) with \( a \neq b \).
- "Kadec-Klee property if for each sequence \( (a_n) \) in M is weak convergence to \( a \) together with \( \|a_n\| \) converges strongly to \( \|a\| \) imply that \( (a_n) \) is strong convergence to a point \( a \in M \) [7]."

The aim of this paper, an iterative scheme for two families of total asymptotically quasi-nonexpansive maps is established. The strong and weak convergence theorems of this scheme for approximation of common fixed points in Banach space by using suitable conditions are established. For this purpose, let us recall the following definitions and lemmas.

Definition (1.1): "A map T is named asymptotically nonexpansive [1] if there is a sequence \( (f_n) \) in \([0, +\infty)\) with \( \lim_{n \to \infty} f_n = 0 \) and \( \|T^n a - T^n b\| \leq (1 + f_n) \|a - b\| \) for all \( a, b \in B, n = 1, 2, \ldots \)"

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www.astesj.com
https://dx.doi.org/10.25046/aj040311
If $F(T) \neq \emptyset$ there is a sequence $(f_n)_{n=1}^\infty$ in $[0, +\infty)$ with $\lim_{n \to \infty} f_n = 0$ and $\|T^n a - a^*\| \leq (1 + f_n)\|a - a^*\|$, for all $a \in B, a^* \in F(T)$ and $n = 1, 2, \ldots$. Therefore, $T$ is named asymptotically quasi-nonexpansive map [10].

**Definition (1.2):** A map $T$ is named total asymptotically nonexpansive map [4] if there are null sequences of positive real numbers $(f_n)_{n=1}^\infty$, $(g_n)_{n=1}^\infty$, $n \geq 1$ and nondecreasing continuous function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that for all $a, b \in B$

$$\|T^n a - T^n b\| \leq \|a - b\| + f_n \|a - b\| + g_n.$$ 

$T$ is named total asymptotically quasi-nonexpansive map if $F(T) \neq \emptyset$, are null sequences of positive real numbers $(f_n)_{n=1}^\infty$, $(g_n)_{n=1}^\infty$, $n \geq 1, \sum_{n=1}^\infty f_n < \infty$ and $\sum_{n=1}^\infty g_n < \infty$, and nondecreasing continuous function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that for all $a, b \in B$ and $a^* \in F(T), a^* \in F(T)$

$$\|T^n a - a^*\| \leq \|a - a^*\| + f_n \|a - a^*\| + g_n.$$ 

If $g_n = 0, \forall n \geq 1, 2, \ldots$ therefore $T$ is asymptotically quasi-nonexpansive map.

**Definition (1.3)[11]:** Let $B$ be a nonempty closed convex subset of a Banach space $M$. A self-map $T$ is named uniformly Lipschitzian if there exists a constant $K > 0$ such that

$$\|T^n a - T^n b\| \leq K\|a - b\|, \forall a, b \in B.$$ 

**Definition (1.4)[12]:** A map $T: B \to M$ is named demi-closed with respect to $b \in M$ if for each sequence $(a_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$, $a_n \to a$ weak convergence to $a$ and $T(a_n)$ is strong convergence to $b$. Hence $a \in B$ and $T(a) = b$. If $(I - T)$ is demiclosed which means if $(a_n)$ is weak convergence to $a$ in $B$ and $(I - T)$ is strong convergence to $0$. Therefore $(I - T)(a) = 0$.

**Note:** Now to explain the relation between the above definitions:

**Nonexpansive $\Rightarrow$ Asymptotically $\Rightarrow$ Total asymptotically nonexpansive $\Rightarrow$ Quasi-$\Rightarrow$ Asymptotically quasi $\Rightarrow$ Total asymptotically quasi-nonexpansive $\Rightarrow$ nonexpansive $\Rightarrow$ quasi - nonexpansive $\Rightarrow$ nonexpansive**

**Lemma (1.5)[13]:** Let $M$ be a uniformly convex Banach space and $0 < L \leq t_n \leq K < 1, \forall n \in N$. Assume that $(a_n)$ and $(b_n)$ are two sequences of $M$ such as:

$$\lim_{n \to \infty} \|a_n\| \leq r, \lim_{n \to \infty} \|b_n\| \leq r \text{ and } \lim_{n \to \infty} \|T^n a_n + (1 - t_n) b_n\| = r$$ 

hold for some $r \geq 0$. Thus $\lim_{n \to \infty} \|a_n - b_n\| = 0$.

**Lemma (1.6)[14]:** Let $(\mu_n)_{n=1}^\infty$, $(\sigma_n)_{n=1}^\infty$ and $(e_n)_{n=1}^\infty$ be sequences of positive numbers satisfying the following inequality:

$$h_{n+1} \leq (1 + \sigma_n)h_n + e_n, \forall n \geq 1$$ 

if $\sum_{n=1}^\infty \sigma_n < \infty$ and $\sum_{n=1}^\infty e_n < \infty$, then $(\mu_n)$ is bounded and $\lim_{n \to \infty} \mu_n$ exists. In additional if, $\lim_{n \to \infty} h_n = 0$ then $\lim_{n \to \infty} \mu_n = 0$.

**Lemma (1.7)[15]:** Let $B$ be a nonempty closed convex subset of a uniformly convex Banach space. Therefore there exists a strictly nondecreasing continuous function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ such as for each Lipschitz map $T: B \to B$ with Lipschitz constant $K$:

$$\|T a + (1 - t) T b - T(\alpha a + \beta b)\| \leq Kf^{-1}(\|a - b\| - \frac{1}{K}\|Ta - Tb\|).$$ 

$\forall a, b \in B$ and $\forall t, \alpha \in [0, 1]$.

**Lemma (1.8)[12]:** "Let $M$ be a uniformly convex Banach space and its dual $M^*$ accomplishing the Kadec-Klee property. Presume that $(a_n)$ bounded sequence in $M$ such as $\lim_{n \to \infty} \|t a_n + (1 - t) b|p_1 - p_2|\|$ exists $\forall t \in [0, 1]$ and $p_1, p_2 \in W(a_n), thus p_1 = p_2$.

2. **Main Results**

Let $B$ be a nonempty closed convex subset of a Banach space $M$ and $T_j, S_j, \forall j = 1, 2, \ldots, k$ be two families of total asymptotically quasi-nonexpansive self-maps. We define the iteration algorithm $(a_n)$ as follows:

$$a_{n+1} \in B$$

$$a_{n+1} = (1 - \alpha_n) S_j a_n + \alpha_n T_j b_n$$

$$b_{n+1} = (1 - \alpha_n) S_n a_n + \alpha_n T_n b_{n+1}$$

$$b_{n+1} = (1 - \alpha_n) S_{n+1} a_n + \alpha_n T_{n+1} b_{n+1}$$

Where $b_0 = a_n$ and $(a_n)_{n=1}^\infty$ are sequences in $[0, 1]$.

**Lemma (2.1):** Let $B$ be a nonempty closed convex subset of a normed space $M$ and $T_j, S_j, j = 1, 2, \ldots, k$ be two family of total asymptotically quasi-nonexpansive self-maps of $B$. Presume that $F(T_j, S_j) \neq \emptyset$ and the sequence $(a_n)$ be as shown in step (1). Then:

i- There are sequences $(u_n)$ and $(v_n)$ in $[0, \infty)$ such as $\sum_{n=1}^\infty u_n < \infty, \sum_{n=1}^\infty v_n < \infty$ and

$$\|a_{n+1} - a^*\| \leq (1 + u_n)\|a_n - a^*\| + v_n + 1, \forall a^* \in F(T_j, S_j) \text{ and } n.$$ 

ii- There exist constants $j_1, j_2 > 0$ such as

$$\|a_{n+1} - a^*\| \leq j_1\|a_n - a^*\| + j_2, \forall a^* \in F(T_j, S_j) \text{ and } n, p = 1, 2, \ldots$$

If there is $Z > 0$ such that $\psi(j_2) \leq Z\lambda_j, j = 1, 2, \ldots, k$.

**Proof:** i- Let $a^* \in F, u_n = \max_{1 \leq j \leq k} f_{j+n}$ and $v_n = \max_{1 \leq j \leq k} j_n$.

Now, we have

$$\|b_n - a^*\| = \|(1 - \alpha_n) S_n a_n + \alpha_n T_n a_n - a^*\|$$

$$\leq (1 - \alpha_n)\|S_n a_n - a^*\| + \alpha_n\|T_n a_n - a^*\|$$

$$\leq (1 - \alpha_n)\|a_n - a^*\| + f_n\|a_n - a^*\| + g_{1n}$$

$$\alpha_n\{\|a_n - a^*\| + f_n\|a_n - a^*\| + g_{1n}\}$$

$$\leq (1 - \alpha_n)\{1 + f_n Z\}\|a_n - a^*\| + (1 - \alpha_n)g_{1n} + \alpha_n(1 + f_n Z)\|a_n - a^*\| + a_n g_{1n}$$

$$\leq (1 + f_n Z)\|a_n - a^*\| + g_{1n} + 1 < u_n\|a_n - a^*\| + v_n$$

Assume that $\|b_{n+1} - a^*\| \leq (1 + u_{n+1})\|a_n - a^*\| + v_{n+1}$

Therefore

$$\|b_{n+1} - a^*\| = \|(1 - \alpha_(j+n+1)) S_{n+1} a_n + \alpha_{j+n+1} T_{j+n+1} b_{j+n+1} - a^*\|$$

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\[ (1 - \alpha_{(j+1)n}) ||S^n_{j+1}a_n - a^*|| \\
+ \alpha_{(j+1)n}||T^n_{j+1}b_j - a^*|| \\
\leq (1 - \alpha_{(j+1)n}) (1 + f_{(j+1)n}Z)||a_n - a^*|| \\
+ (1 - \alpha_{(j+1)n}) g_{(j+1)n} \\
+ \alpha_{(j+1)n} (1 + f_{(j+1)n}Z)||b_j - a^*|| \\
\leq (1 - \alpha_{(j+1)n}) (1 + u_n)||a_n - a^*|| + v_n \\
+ \alpha_{(j+1)n} (1 + u_n)(1 + u_n)/||a_n - a^*|| \\
+ \alpha_{(j+1)n} (1 + u_n)v_n/ \\
\leq (1 + u_n)^j/||a_n - a^*|| + v_n^{j+1} \]

Thus, by induction, we obtain
\[ \|b_j - a^*\| \leq (1 + u_n)^j/||a_n - a^*|| + v_n^{j+1} \] for all \( j = 1, 2, ..., k \).

Now, by (2), we get
\[ \|a_{n+1} - a^*\| \leq (1 - \alpha_n)||S^na_n - a^*|| + \alpha_n||T^nb_j - a^*|| \\
\leq (1 - \alpha_n) (1 + u_n)||a_n - a^*|| + v_n \\
+ \alpha_n (1 + u_n)(1 + u_n)/||a_n - a^*|| \\
+ \alpha_n (1 + u_n)v_n/ \\
\leq (1 + u_n)^{j+1}/||a_n - a^*|| + v_n^{j+1} \]

ii- By using part (i), we get
\[ \|a_{n+p} - a^*\| \leq (1 + u_{n+p-1})^{j+1}/||a_{n+p-1} - a^*|| + v_n^{j+1} \]
\[ \leq e^{(1+u_{n+p-1})^{j+1}/||a_{n+p-1} - a^*||} + v_n^{j+1} \]
\[ \leq e^{(j+1)u_{n+p-1}/||a_{n+p-1} - a^*||} + e^{(j+1)v_n^{j+1}} \]
\[ \leq (j+1)\Sigma_{k=1}^{n+p-1} u_k/||a_n - a^*|| + e^{(j+1)\Sigma_{k=1}^{n+p-1} v_k} \]
\[ \leq J_1||a_n - a^*|| + J_2. \]

Setting \( J_1 = e^{(j+1)\Sigma_{k=1}^{n+p-1} u_k} \) and \( J_2 = e^{(j+1)\Sigma_{k=1}^{n+p-1} v_k} \).

Lemma (2.2): Let \( B \) be a nonempty closed convex subset of a normed space \( M \) and \( T_j, S_j, j = 1, 2, ..., k \) be two families of total asymptotically quasi-nonexpansive self-maps of \( B \). Assume \( F(T_j, S_j) \neq \emptyset \) and \( (a_n) \) be as shown in step (1). Therefore, let \( \lim_{n \to \infty} ||a_n - a^*|| \) exists for all \( a^* \in F(T_j, S_j) \).

Proof: By Lemma (2.1),\( i) \)
\[ ||a_{n+1} - a^*|| \leq (1 + u_n)^{j+1}/||a_{n+p-1} - a^*|| + v_n^{j+1} \]
and \( \Sigma_{n=1}^{\infty} u_n < \infty, \Sigma_{n=1}^{\infty} v_n < \infty \). So by Lemma (1.6.i), we get
\[ \lim_{n \to \infty} ||a_{n+1} - a^*|| \] exists for all \( a^* \in F(T_j, S_j) \).

Lemma (2.3): Let \( B \) be a nonempty closed convex subset of a Banach space \( M \) and \( T_j, S_j, j = 1, 2, ..., k \) be two families of Lipschitzian and total asymptotically quasi-nonexpansive self-maps of \( B \). Let \( (a_n) \) be as shown in step (1). Therefore, for all \( a_n^2, a_n^2 \in F(T_j, S_j) \), the limit
\[ \lim_{n \to \infty} ||ta_n + (1-t)a_n^2 - a_2^2|| \] exists for all \( t \in [0, 1] \).

If there is \( Z > 0 \) such that \( \psi(\lambda) \leq Z\lambda, j = 1, 2, ..., k \).

Proof: By using Lemma (1.6), we have \( \lim_{n \to \infty} ||a_n - a^*|| \) exists \( \forall a^* \in F(T_j, S_j) \) and \( (a_n) \) is bounded. Let
\[ \gamma_n(t) = ||ta_n + (1-t)a_n^2 - a_2^2||. \text{ For } t \in [0, 1]. \]
Therefore, \( \lim_{n \to \infty} a_n(0) = ||a_n^2 - a_2^2|| \) and \( \lim_{n \to \infty} a_n(1) = ||a_n - a_2^2|| \) exist by Lemma (2.2).

Then, for \( t \in [0, 1] \) and for all \( a \in A \), we define the map \( R_n : B \to B \) by:
\[ b_{1n} = (1 - \alpha_{n}) S^n a_n + \alpha_{n} T^n b_{2n} \\
\]
\[ b_{2n} = (1 - \alpha_{n}) S^n a_n + \alpha_{n} T^n b_{2n} \]

Now,
\[ ||R_n a - R_n c|| \leq (1 - \alpha_n)||S^n a - S^n c|| + \alpha_n||T^n b_j - T^n d_j|| \\
\leq (1 - \alpha_n)(1 + f_{n}Z)||a - c|| + (1 - \alpha_n)g_{n} \\
+ \alpha_n(1 + f_{n}Z)||b_j - d_j|| + \alpha_n g_{n} \]
\[ \leq (1 + u_n)||a - c|| + v_n \\
+ (1 + f_{n}Z)||a - b|| + g_n \]
with \( \Sigma_{n=1}^{\infty} u_n < \infty, \Sigma_{n=1}^{\infty} v_n < \infty \) and \( s_n = 1 + u_n \), it follows that \( s_n \to 1 \) as \( n \to \infty \).

Setting \( W_{n,m} = R_{n+m-1} R_{n+m-2} ... R_n \)
and \( b_{n,m} = ||W_{n,m} (ta_n + (1-t)a_n^2 - (tW_{n,m} a_n + (1-t)a_n^2))||. \)

Thus,
\[ ||W_{n,m} a - W_{n,m} b|| \leq ||R_{n+m-1} ... R_{n} (a) - R_{n+m-2} ... R_{n} (b)|| + \sum_{n=m}^{n=m-1} v_j \]
\[ = A_n||a - c|| + \sum_{n=m}^{n=m-1} v_j \]
for all \( a, c \in B \), where \( A_n = \prod_{j=m}^{n=m-1} s_j, W_{n,m} a_n = a_{n+m} \) and \( a^* = a^* \) for all \( a^* \in F(T_i, S_i) \).

Hence,
\[ \gamma_n(t) = ||ta_n + (1-t)a_n^2 - a_2^2|| \\
\]
\[ = ||tW_{n,m} a_n + (1-t)a_n^2 + W_{n,m} (ta_n + (1-t)a_n^2) - a_2^2|| \\
+ ||a_n^2 - a^*|| \]
\[ \leq W_{n,m} (t a_n + (1-t)a_n^2) - a_2^2 \]
\[ \leq b_{n,m} + ||W_{n,m} (ta_n + (1-t)a_n^2) - a_2^2|| \]
\[ \leq b_{n,m} + A_n\gamma_n(t) + \sum_{n=m}^{n=m-1} v_j \]
By using Lemma (1.7), we have
\[ b_{n,m} \leq K f^{-1}(||a_n - a^*|| - \frac{1}{K} ||W_{n,m} a_n - W_{n,m} a^*||) \]
\[ \leq K f^{-1}(||a_n - a^*|| - \frac{1}{K} (||a_{n+m} - a^*|| - ||W_{n,m} a_n - a^*||)) \]
and \( (b_{n,m}) \) converges uniformly to zero. Since \( \lim_{n \to \infty} A_n = 1 \) and \( \lim_{n \to \infty} v_n = 0 \), we get
\[ \lim_{n \to \infty} \sup_{y_n} \gamma_n \leq \lim_{n \to \infty} b_{n,m} + \lim_{n \to \infty} \inf_n \gamma_n(t) = \lim_{n \to \infty} \inf_n \gamma_n(t) \]

Thus, \( \lim_{n \to \infty} \gamma_n(t) \) exists for all \( t \in [0, 1] \).
Theorem (2.4): Let $B$ be a nonempty closed convex subset of a uniformly convex Banach space $M$, $T_j, S_j, \forall j = 1, 2, ..., k$ be two families of Lipschitzian and total asymptotically quasi-nonexpansive self-maps of $B$ and the sequence $(a_n)$ be as shown in step (1). If $Z > 0$ such that $\psi((T_j)^j) \subseteq Z \lambda_j, j = 1, 2, ..., k$. Then, $\lim_{n \to \infty} \|T^n_j a_n - a_n\| = 0 = \lim_{n \to \infty} \|T^n_j S_n a_n - a_n\|, \forall j$.

Proof: By Lemma (2.2), $\lim_{n \to \infty} \|a_n - a^*\|$ exists. Assume that $\lim_{n \to \infty} \|a_n - a^*\| = e, \forall e \geq 0$.

Now, suppose $e = 0$. The proof is straightforward.

Next, $\lim_{n \to \infty} \|T^n_j S_n a_n - T^n_j b_j\| = 0$.

Then, $\lim_{n \to \infty} \|S^n_j a_n - T^n_j a_n\| = 0$.

Since $Z > 0$, $\sum_{n=1}^{\infty} f_j < \infty$ and $\sum_{n=1}^{\infty} g_j < \infty$, hence $\lim_{n \to \infty} \|S^n_j a_n - T^n_j b_j - a_n\| = 0$.

And $\lim_{n \to \infty} \|S^n_j a_n - T^n_j a_n\| = 0$.

Theorem (2.5): Let $B$ be a nonempty closed convex subset of a Banach space $T_j, S_j, j = 1, 2, ..., k$ be two families of total asymptotically quasi-nonexpansive self-maps of $B$. Assume that $F(T_j, S_j) \neq \emptyset$ and $(a_n)$ be as shown in step (1) is strong convergence to a common fixed point of $T_j$ and $S_j$ iff

$\lim_{n \to \infty} \inf d(a_n, F) = 0$, where $d(a, F) = \inf_{a \in F} \|a - a^*\|$.

Proof: To show $\lim_{n \to \infty} \inf d(a_n, F) = 0$ implies that $(a_n)$ is strong convergence to a common fixed point of $T_j, S_j, j = 1, 2, ..., k$, since by (2)
Theorem (2.6): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, j = 1, 2, \ldots, k$ be two families of Lipschitzian and total asymptotically quasi-nonexpansive self-maps of B. If M accomplishes Opial’s condition and the maps $I - T_j$ and $I - S_j, j = 1, 2, \ldots, k$ are demiclosed to zero, therefore $(a_n)$ be as shown in step (1) is weak convergence to a common fixed point of $T_j$ and $S_j, j = 1, 2, \ldots, k$.

Proof: Let $a^* \in F(T_j, S_j)$. By Lemma (2.2), $\lim_{n \to \infty} ||a_n - a^*||$ exists.

By Theorem (2.4), we have
\[
\lim_{n \to \infty} ||T_j^n a_n - a_n|| = 0 = \lim_{n \to \infty} ||S_j^n a_n - a_n||
\]
for $j = 1, 2, \ldots, k$. Since by the supposition the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \ldots, k$ are demiclosed to zero, therefore $T_j a^* = a^*$ and $S_j a^* = a^*$, that means $a^* \in F(T_j, S_j)$.

Next, to prove $(a_n)$ converges weakly to $a^*$. Assume there is another subsequence $(a_{n_k})$ of $(a_n)$ is weak convergence to $b^* \in F(T_j, S_j)$ and $a^* \neq b^*$. By using the same argument as above we can show that $b^* \in F(T_j, S_j)$.

Now, to prove the uniqueness, assume $a^* \neq b^*$. Therefore, by using Opial’s condition, we obtain:
\[
\lim_{n \to \infty} ||a_n - a^*|| = \lim_{n \to \infty} ||a_{n_j} - a^*||
\leq \lim_{n \to \infty} ||a_{n_j} - b^*||
= \lim_{n \to \infty} ||a_{n_l} - b^*||
< \lim_{n \to \infty} ||a_{n_l} - a^*||
= \lim_{n \to \infty} ||a_{n_l} - b^*||
\]
This is contradiction, therefore $a^* \neq b^*$. Hence, $(a_n)$ is weak convergence to $a^*$.

Theorem (2.7): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, j = 1, 2, \ldots, k$ be two families of Lipschitzian and total asymptotically quasi-nonexpansive self-maps of B. If the dual space $M^*$ of M has the Kadec-klee property and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \ldots, k$ are demi-closed to zero, therefore, $(a_n)$ be as shown in step (1) is weak convergence to a common fixed point of $T_j$ and $S_j$.

Proof: As showed by Lemma (2.2), $\lim_{n \to \infty} ||a_n - a||$ exists.

Since $(a_n)$ is bounded in B and M is reflexive. Therefore, there exists a subsequence $(a_{n_l})$ of $(a_n)$ which is weak convergence to a point $a^* \in B$. By Theorem (2.4)
\[
\lim_{n \to \infty} ||T_j^n a_n - a_n|| = 0 = \lim_{n \to \infty} ||S_j^n a_n - a_n||
\]
for $j = 1, 2, \ldots, k$.

Since by the supposition the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \ldots, k$ are demi-closed to zero. Thus, $a^* \in F(T_j, S_j)$. Now, to prove $(a_n)$ is weak convergence to a point $a^*$. Assume that $(a_{n_k})$ is another subsequence of $(a_n)$ that is weak convergence to a point $b^* \in F(T_j, S_j)$. By using the same argument as above, we obtain $b^* \in F(T_j, S_j)$.

Therefore, by Lemma (2.3) $\lim_{n \to \infty} ||t a_n + (1 - t) a^* - b^*||$ exists for all $t \in [0, 1]$.

By Lemma (1.8) $a^* = b^*$. As a result, the sequence $(a_n)$ is weak convergence to the point $a^* \in F(T_j, S_j)$.

Corollary (2.8): Let B be a nonempty closed convex subset of a Banach space and $T_j, S_j, j = 1, 2, \ldots, k$ be two families of total asymptotically nonexpansive self-maps of B. Presume that $F(T_j, S_j) \neq \emptyset$ and $\sum_{i=1}^{\infty} f_i < \infty, \sum_{i=1}^{\infty} g_i < \infty$.

Presume that $(a_n)$ be as shown in step (1) is strong convergence to a a common fixed point of $T_j$ and $S_j$ iff
\[
\lim_{n \to \infty} inf \{d(a_n, F) = 0\}, \text{where} \ d(a, F) = inf_{a \in F} ||a - a^*||.
\]

Corollary (2.9): Let $T_j, S_j, B, f_j$ and $g_j, j = 1, 2, \ldots, k$ be as in corollary (2.8). Therefore $(a_n)$ be as shown in step (1) is strong convergence to $a^* \in F(T_j, S_j)$ iff $(a_n)$ of $(a_n)$ that converges to $a^*$.

Corollary (2.10): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, \forall j = 1, 2, \ldots, k$ be two families of Lipschitzian and total asymptotically nonexpansive self-maps of B. If the dual space $M^*$ of M has the Kadec-klee property and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \ldots, k$ are demi-closed to zero, therefore, $(a_n)$ be as shown in step (1) is weak convergence to a common fixed point of $T_j$ and $S_j$.

Corollary (2.11): Let B be a nonempty closed convex subset of a Banach space and $T_j, S_j, \forall j = 1, 2, \ldots, k$ be two families of total asymptotically nonexpansive self-maps of B. If M accomplishes Opial’s condition and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \ldots, k$ are demi-closed to zero, therefore, $(a_n)$ be as shown in step (1) is weak convergence to a common fixed point of $T_j$ and $S_j$.

Corollary (2.12): Let B be a nonempty closed convex subset of a Banach space and $T_j, S_j, \forall j = 1, 2, \ldots, k$ be two families of asymptotically quasi-nonexpansive self-maps of B. Presume that $F(T_j, S_j) \neq \emptyset$ and $\sum_{i=1}^{\infty} f_i < \infty$. Presume that $(a_n)$ be as shown in step (1) is strong convergence to a common fixed point of $T_j$ and $S_j$ iff
\[
\lim_{n \to \infty} inf d(a_n, F) = 0, \text{where} \ d(a, F) = inf_{a \in F} ||a - a^*||.
\]

Corollary (2.13): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, \forall j = 1, 2, \ldots, k$ be two families of Lipschitzian and asymptotically quasi-nonexpansive self-maps of B. If the dual space $M^*$ of M has the Kadec-klee property and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \ldots, k$ are demi-closed to zero, therefore, $(a_n)$ be as shown in step (1) is weak convergence to a common fixed point of $T_j$ and $S_j$.

Corollary (2.14): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, \forall j = 1, 2, \ldots, k$ be two families of Lipschitzian and asymptotically quasi-nonexpansive selfmaps of B. If M accomplishes Opial’s condition and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \ldots, k$ are demi-closed to zero, therefore, $(a_n)$ be as shown in step (1) is weak convergence to a common fixed point of $T_j$ and $S_j$. We illustrate our results by the following
Example (3.1): Let \( T_j, S_j : \mathcal{R} \rightarrow \mathcal{R}, \forall j = 1, 2, ..., k \) be two maps such that \( T_j a = \frac{2a}{3} \) and \( S_j a = \frac{46a}{21}, \forall a \in \mathcal{R} \). Choose \( \alpha_n = \frac{n}{4(n+1)} \forall n \) with initial value \( a_1 = 15 \). Let \( (a_n) \) be the sequence. According to Table 1 and Figure 1, it eases to sight that \( (a_n) \) converges to the fixed point \( a^* = 0 \).

Table 1: Numerical results corresponding to \( a_1 = 15 \) for 36 steps.

<table>
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<tr>
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<th>Iteration (1)</th>
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</table>

Figure 1. Convergence behavior corresponding to \( a_1 = 15 \) for 36 steps.

4. Conclusion

We study the strongly and weakly convergence of new type of finite-step iteration processes under total asymptotically quasi-nonexpansive maps, see Theorems (2.4)-(2.6). Our results are generalizing and unifying the results of others who have been referred to in the references.

5. Open Problem

Recently, S.S. Abed has been defined as the following type of generalizations of total asymptotically quasi-nonexpansive[16]: Let \( \mathcal{A} \) be a subset of real Banach space a set–valued map \( G: \mathcal{A} \rightarrow 2^{\mathcal{A}} \) is called the general asymptotic set-valued if for each \( x \in \mathcal{A} \) there exists null non- negative real sequences \( \{a_n\} \) and \( \{b_n\} \) such that

\[
\mathcal{D}(G^n x, G^n y) \leq \|x - w\| + a_n \mu(\|x - w\|) + b_n
\]

for any \( y \in \mathcal{A} , w \in G^n y \) and \( \mu: R^+ \rightarrow R^+ \) with \( \mu(0) = 0 \).

One can study convergence theorems in (1) and in [17, theorem (11)] for families of general asymptotic set-valued maps. As well as possible to demonstrate new results in the case of other spaces as a modular space [18].

Acknowledgment

The authors wish to express his thanks to the referees for their helpful advice.

References


